MUIC APPLIED MATH SEMINAR

abc Conjecture and Its Applications

Chatchawan Panraksa

Mahidol University International College

1 May 2019

$1+1=2$

1+1=2

$6 + 19 = 25 = 5^2$

1+1=2
$6+19=25=5^2$
$3^2+4^2=5^2$

1+1=2

$6 + 19 = 25 = 5^2$

$3^2 + 4^2 = 5^2$ Pythagorean tuple

1+1=2
$6+19=25=5^2$
$3^2+4^2=5^2$ Pythagorean tuple
$1^3+12^3=1729=9^3+10^3$

1+1=2

$6+19=25=5^2$

$3^2+4^2=5^2$ Pythagorean tuple

$1^3+12^3=1729=9^3+10^3$taxicab number

1+1=2

$6+19=25=5^2$

$3^2+4^2=5^2$ Pythagorean tuple

$1^3+12^3=1729=9^3+10^3$taxicab number

$1729=7\times 13\times 19$

1+1=2

$6+19=25=5^2$

$3^2+4^2=5^2$ Pythagorean tuple

$1^3+12^3=1729=9^3+10^3$taxicab number

$1729=7\times 13\times 19$

$2^8+3^7=2443=7\times 349$

$\vdots$

1+1=2

$6+19=25=5^2$

$3^2+4^2=5^2$ Pythagorean tuple

$1^3+12^3=1729=9^3+10^3$taxicab number

$1729=7\times 13\times 19$

$2^8+3^7=2443=7\times 349$

$\vdots$

$x^n+y^n=z^n, n>2$ Fermat's Last Theorem

$\vdots$

$$a+b=c$$

Definition

A **radical** of a positive integer is a product of its distinct prime factors.
NOTATION: $n = p_1^{k_1} p_1^{k_1} \ldots p_m^{k_m}$
radical of n is $rad(n) = p_1 p_2 \ldots p_m$.
$rad(1) := 1$

Definition

A **radical** of a positive integer is a product of its distinct prime factors.
NOTATION: $n = p_1^{k_1} p_1^{k_1} \dots p_m^{k_m}$
radical of n is $rad(n) = p_1 p_2 \dots p_m$.
$rad(1) := 1$

Example

$rad(5) = 5$

Definition

A **radical** of a positive integer is a product of its distinct prime factors.
NOTATION: $n = p_1^{k_1} p_1^{k_1} \dots p_m^{k_m}$
radical of n is $rad(n) = p_1 p_2 \dots p_m$.
$rad(1) := 1$

Example

$rad(5) = 5$
$rad(345744)$

Definition

A **radical** of a positive integer is a product of its distinct prime factors.
NOTATION: $n = p_1^{k_1} p_1^{k_1} \dots p_m^{k_m}$
radical of n is $rad(n) = p_1 p_2 \dots p_m$.
$rad(1) := 1$

Example

$rad(5) = 5$
$rad(345744) = rad(2^4 \cdot 3^2 \cdot 7^4)$

Definition

A **radical** of a positive integer is a product of its distinct prime factors.
NOTATION: $n = p_1^{k_1} p_1^{k_1} \dots p_m^{k_m}$
radical of n is $rad(n) = p_1 p_2 \dots p_m$.
$rad(1) := 1$

Example

$rad(5) = 5$
$rad(345744) = rad(2^4 \cdot 3^2 \cdot 7^4) = 2 \cdot 3 \cdot 7$

Definition

A **radical** of a positive integer is a product of its distinct prime factors.
NOTATION: $n = p_1^{k_1} p_1^{k_1} \dots p_m^{k_m}$
radical of n is $rad(n) = p_1 p_2 \dots p_m$.
$rad(1) := 1$

Example

$rad(5) = 5$
$rad(345744) = rad(2^4 \cdot 3^2 \cdot 7^4) = 2 \cdot 3 \cdot 7 = 42$

Definition

A **radical** of a positive integer is a product of its distinct prime factors.
NOTATION: $n = p_1^{k_1} p_1^{k_1} \dots p_m^{k_m}$
radical of n is $rad(n) = p_1 p_2 \dots p_m$.
$rad(1) := 1$

Example

$rad(5) = 5$
$rad(345744) = rad(2^4 \cdot 3^2 \cdot 7^4) = 2 \cdot 3 \cdot 7 = 42$
$rad(1868347265625)$

Definition

A **radical** of a positive integer is a product of its distinct prime factors.
NOTATION: $n = p_1^{k_1} p_1^{k_1} \dots p_m^{k_m}$
radical of n is $rad(n) = p_1 p_2 \dots p_m$.
$rad(1) := 1$

Example

$rad(5) = 5$
$rad(345744) = rad(2^4 \cdot 3^2 \cdot 7^4) = 2 \cdot 3 \cdot 7 = 42$
$rad(1868347265625) = rad(3^{14} \cdot 5^8)$

Definition

A **radical** of a positive integer is a product of its distinct prime factors.
NOTATION: $n = p_1^{k_1} p_1^{k_1} \dots p_m^{k_m}$
radical of n is $rad(n) = p_1 p_2 \dots p_m$.
$rad(1) := 1$

Example

$rad(5) = 5$
$rad(345744) = rad(2^4 \cdot 3^2 \cdot 7^4) = 2 \cdot 3 \cdot 7 = 42$
$rad(1868347265625) = rad(3^{14} \cdot 5^8) = 3 \cdot 5$

Definition

A **radical** of a positive integer is a product of its distinct prime factors.
NOTATION: $n = p_1^{k_1} p_1^{k_1} \dots p_m^{k_m}$
radical of n is $rad(n) = p_1 p_2 \dots p_m$.
$rad(1) := 1$

Example

$rad(5) = 5$
$rad(345744) = rad(2^4 \cdot 3^2 \cdot 7^4) = 2 \cdot 3 \cdot 7 = 42$
$rad(1868347265625) = rad(3^{14} \cdot 5^8) = 3 \cdot 5 = 15$

$a + b = c$, c and $rad(abc)$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$

$a+b=c$, c and $rad(abc)$

$1+1=\mathbf{2}, rad(1\cdot 1\cdot 2)=\mathbf{2}$
$6+19=\mathbf{25}, rad(6\cdot 19\cdot 25)=2\cdot 3\cdot 19\cdot 5=\mathbf{570}$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$
$6 + 19 = \mathbf{25}, rad(6 \cdot 19 \cdot 25) = 2 \cdot 3 \cdot 19 \cdot 5 = \mathbf{570}$
$3^2 + 4^2 = 5^2 = \mathbf{25}, rad(3^2 \cdot 4^2 \cdot 5^2) = 3 \cdot 2 \cdot 5 = \mathbf{30}$

$a + b = c$, c and $rad(abc)$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$
$6 + 19 = \mathbf{25}, rad(6 \cdot 19 \cdot 25) = 2 \cdot 3 \cdot 19 \cdot 5 = \mathbf{570}$
$3^2 + 4^2 = 5^2 = \mathbf{25}, rad(3^2 \cdot 4^2 \cdot 5^2) = 3 \cdot 2 \cdot 5 = \mathbf{30}$
$1729 = 7 \cdot 13 \cdot 19$

$a + b = c$, c and $rad(abc)$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$
$6 + 19 = \mathbf{25}, rad(6 \cdot 19 \cdot 25) = 2 \cdot 3 \cdot 19 \cdot 5 = \mathbf{570}$
$3^2 + 4^2 = 5^2 = \mathbf{25}, rad(3^2 \cdot 4^2 \cdot 5^2) = 3 \cdot 2 \cdot 5 = \mathbf{30}$
$1729 = 7 \cdot 13 \cdot 19$
$1^3 + 12^3$

$a + b = c$, c and $rad(abc)$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$
$6 + 19 = \mathbf{25}, rad(6 \cdot 19 \cdot 25) = 2 \cdot 3 \cdot 19 \cdot 5 = \mathbf{570}$
$3^2 + 4^2 = 5^2 = \mathbf{25}, rad(3^2 \cdot 4^2 \cdot 5^2) = 3 \cdot 2 \cdot 5 = \mathbf{30}$
$1729 = 7 \cdot 13 \cdot 19$
$1^3 + 12^3 = \mathbf{1729} = 9^3 + 10^3$

$a + b = c$, c and $rad(abc)$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$
$6 + 19 = \mathbf{25}, rad(6 \cdot 19 \cdot 25) = 2 \cdot 3 \cdot 19 \cdot 5 = \mathbf{570}$
$3^2 + 4^2 = 5^2 = \mathbf{25}, rad(3^2 \cdot 4^2 \cdot 5^2) = 3 \cdot 2 \cdot 5 = \mathbf{30}$
$1729 = 7 \cdot 13 \cdot 19$
$1^3 + 12^3 = \mathbf{1729} = 9^3 + 10^3, rad(1^3 \cdot 12^3 \cdot 1729)$

$a + b = c$, c and $rad(abc)$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$
$6 + 19 = \mathbf{25}, rad(6 \cdot 19 \cdot 25) = 2 \cdot 3 \cdot 19 \cdot 5 = \mathbf{570}$
$3^2 + 4^2 = 5^2 = \mathbf{25}, rad(3^2 \cdot 4^2 \cdot 5^2) = 3 \cdot 2 \cdot 5 = \mathbf{30}$
$1729 = 7 \cdot 13 \cdot 19$
$1^3 + 12^3 = \mathbf{1729} = 9^3 + 10^3, rad(1^3 \cdot 12^3 \cdot 1729) = 12 \cdot 7 \cdot 13 \cdot 19 = \mathbf{20748}$

$a + b = c$, c and $rad(abc)$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$
$6 + 19 = \mathbf{25}, rad(6 \cdot 19 \cdot 25) = 2 \cdot 3 \cdot 19 \cdot 5 = \mathbf{570}$
$3^2 + 4^2 = 5^2 = \mathbf{25}, rad(3^2 \cdot 4^2 \cdot 5^2) = 3 \cdot 2 \cdot 5 = \mathbf{30}$
$1729 = 7 \cdot 13 \cdot 19$
$1^3 + 12^3 = \mathbf{1729} = 9^3 + 10^3, rad(1^3 \cdot 12^3 \cdot 1729) = 12 \cdot 7 \cdot 13 \cdot 19 = \mathbf{20748}$
$rad(9^3 \cdot 10^3 \cdot 1729) = 3 \cdot 2 \cdot 5 \cdot 7 \cdot 13 \cdot 19 = \mathbf{51870}$

$a + b = c$, c and $rad(abc)$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$
$6 + 19 = \mathbf{25}, rad(6 \cdot 19 \cdot 25) = 2 \cdot 3 \cdot 19 \cdot 5 = \mathbf{570}$
$3^2 + 4^2 = 5^2 = \mathbf{25}, rad(3^2 \cdot 4^2 \cdot 5^2) = 3 \cdot 2 \cdot 5 = \mathbf{30}$
$1729 = 7 \cdot 13 \cdot 19$
$1^3 + 12^3 = \mathbf{1729} = 9^3 + 10^3, rad(1^3 \cdot 12^3 \cdot 1729) = 12 \cdot 7 \cdot 13 \cdot 19 = \mathbf{20748}$
$rad(9^3 \cdot 10^3 \cdot 1729) = 3 \cdot 2 \cdot 5 \cdot 7 \cdot 13 \cdot 19 = \mathbf{51870}$
$2^8 + 3^7 = \mathbf{2443} = 7 \times 349, rad(2^8 \cdot 3^7 \cdot 2443) = 2 \cdot 3 \cdot 7 \cdot 349 = \mathbf{14658}$

$a + b = c$, c and $rad(abc)$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$
$6 + 19 = \mathbf{25}, rad(6 \cdot 19 \cdot 25) = 2 \cdot 3 \cdot 19 \cdot 5 = \mathbf{570}$
$3^2 + 4^2 = 5^2 = \mathbf{25}, rad(3^2 \cdot 4^2 \cdot 5^2) = 3 \cdot 2 \cdot 5 = \mathbf{30}$
$1729 = 7 \cdot 13 \cdot 19$
$1^3 + 12^3 = \mathbf{1729} = 9^3 + 10^3, rad(1^3 \cdot 12^3 \cdot 1729) = 12 \cdot 7 \cdot 13 \cdot 19 = \mathbf{20748}$
$rad(9^3 \cdot 10^3 \cdot 1729) = 3 \cdot 2 \cdot 5 \cdot 7 \cdot 13 \cdot 19 = \mathbf{51870}$
$2^8 + 3^7 = \mathbf{2443} = 7 \times 349, rad(2^8 \cdot 3^7 \cdot 2443) = 2 \cdot 3 \cdot 7 \cdot 349 = \mathbf{14658}$
$x^7 + y^7 = \mathbf{z^7}$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$
$6 + 19 = \mathbf{25}, rad(6 \cdot 19 \cdot 25) = 2 \cdot 3 \cdot 19 \cdot 5 = \mathbf{570}$
$3^2 + 4^2 = 5^2 = \mathbf{25}, rad(3^2 \cdot 4^2 \cdot 5^2) = 3 \cdot 2 \cdot 5 = \mathbf{30}$
$1729 = 7 \cdot 13 \cdot 19$
$1^3 + 12^3 = \mathbf{1729} = 9^3 + 10^3, rad(1^3 \cdot 12^3 \cdot 1729) = 12 \cdot 7 \cdot 13 \cdot 19 = \mathbf{20748}$
$rad(9^3 \cdot 10^3 \cdot 1729) = 3 \cdot 2 \cdot 5 \cdot 7 \cdot 13 \cdot 19 = \mathbf{51870}$
$2^8 + 3^7 = \mathbf{2443} = 7 \times 349, rad(2^8 \cdot 3^7 \cdot 2443) = 2 \cdot 3 \cdot 7 \cdot 349 = \mathbf{14658}$
$x^7 + y^7 = \mathbf{z^7}, rad(x^7 y^7 z^7)$

$a+b=c$, c and $rad(abc)$

$1+1=\mathbf{2}, rad(1\cdot 1\cdot 2)=\mathbf{2}$
$6+19=\mathbf{25}, rad(6\cdot 19\cdot 25)=2\cdot 3\cdot 19\cdot 5=\mathbf{570}$
$3^2+4^2=5^2=\mathbf{25}, rad(3^2\cdot 4^2\cdot 5^2)=3\cdot 2\cdot 5=\mathbf{30}$
$1729=7\cdot 13\cdot 19$
$1^3+12^3=\mathbf{1729}=9^3+10^3, rad(1^3\cdot 12^3\cdot 1729)=12\cdot 7\cdot 13\cdot 19=\mathbf{20748}$
$rad(9^3\cdot 10^3\cdot 1729)=3\cdot 2\cdot 5\cdot 7\cdot 13\cdot 19=\mathbf{51870}$
$2^8+3^7=\mathbf{2443}=7\times 349, rad(2^8\cdot 3^7\cdot 2443)=2\cdot 3\cdot 7\cdot 349=\mathbf{14658}$
$x^7+y^7=\mathbf{z^7}, rad(x^7y^7z^7)<\mathbf{xyz}$

$a+b=c$, c and $rad(abc)$

$1+1=\mathbf{2}, rad(1\cdot 1\cdot 2)=\mathbf{2}$
$6+19=\mathbf{25}, rad(6\cdot 19\cdot 25)=2\cdot 3\cdot 19\cdot 5=\mathbf{570}$
$3^2+4^2=5^2=\mathbf{25}, rad(3^2\cdot 4^2\cdot 5^2)=3\cdot 2\cdot 5=\mathbf{30}$
$1729=7\cdot 13\cdot 19$
$1^3+12^3=\mathbf{1729}=9^3+10^3, rad(1^3\cdot 12^3\cdot 1729)=12\cdot 7\cdot 13\cdot 19=\mathbf{20748}$
$rad(9^3\cdot 10^3\cdot 1729)=3\cdot 2\cdot 5\cdot 7\cdot 13\cdot 19=\mathbf{51870}$
$2^8+3^7=\mathbf{2443}=7\times 349, rad(2^8\cdot 3^7\cdot 2443)=2\cdot 3\cdot 7\cdot 349=\mathbf{14658}$
$x^7+y^7=\mathbf{z^7}, rad(x^7y^7z^7)<\mathbf{xyz}<z^3???$

$a + b = c$, c and $rad(abc)$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$

$6 + 19 = \mathbf{25}, rad(6 \cdot 19 \cdot 25) = 2 \cdot 3 \cdot 19 \cdot 5 = \mathbf{570}$

$3^2 + 4^2 = 5^2 = \mathbf{25}, rad(3^2 \cdot 4^2 \cdot 5^2) = 3 \cdot 2 \cdot 5 = \mathbf{30}$

$1729 = 7 \cdot 13 \cdot 19$

$1^3 + 12^3 = \mathbf{1729} = 9^3 + 10^3, rad(1^3 \cdot 12^3 \cdot 1729) = 12 \cdot 7 \cdot 13 \cdot 19 = \mathbf{20748}$

$rad(9^3 \cdot 10^3 \cdot 1729) = 3 \cdot 2 \cdot 5 \cdot 7 \cdot 13 \cdot 19 = \mathbf{51870}$

$2^8 + 3^7 = \mathbf{2443} = 7 \times 349, rad(2^8 \cdot 3^7 \cdot 2443) = 2 \cdot 3 \cdot 7 \cdot 349 = \mathbf{14658}$

$x^7 + y^7 = \mathbf{z^7}, rad(x^7 y^7 z^7) < \mathbf{xyz} < z^3$???

Observation: $c \leq rad(abc)$

$1 + 1 = \mathbf{2}, rad(1 \cdot 1 \cdot 2) = \mathbf{2}$
$6 + 19 = \mathbf{25}, rad(6 \cdot 19 \cdot 25) = 2 \cdot 3 \cdot 19 \cdot 5 = \mathbf{570}$
$3^2 + 4^2 = 5^2 = \mathbf{25}, rad(3^2 \cdot 4^2 \cdot 5^2) = 3 \cdot 2 \cdot 5 = \mathbf{30}$
$1729 = 7 \cdot 13 \cdot 19$
$1^3 + 12^3 = \mathbf{1729} = 9^3 + 10^3, rad(1^3 \cdot 12^3 \cdot 1729) = 12 \cdot 7 \cdot 13 \cdot 19 = \mathbf{20748}$
$rad(9^3 \cdot 10^3 \cdot 1729) = 3 \cdot 2 \cdot 5 \cdot 7 \cdot 13 \cdot 19 = \mathbf{51870}$
$2^8 + 3^7 = \mathbf{2443} = 7 \times 349, rad(2^8 \cdot 3^7 \cdot 2443) = 2 \cdot 3 \cdot 7 \cdot 349 = \mathbf{14658}$
$x^7 + y^7 = \mathbf{z^7}, rad(x^7 y^7 z^7) < \mathbf{xyz} < z^3$???
Observation: $c \leq rad(abc)$
"$rad(abc)$ should not be small compared to c."

$c \leq rad(abc)$?

$1 + 63 = \mathbf{64}$

$c \leq rad(abc)$?

$1 + 63 = \mathbf{64}, rad(1 \cdot 63 \cdot 64) = 3 \cdot 7 \cdot 2 = \mathbf{42}$

$c \leq rad(abc)$?

$1 + 63 = \mathbf{64}, rad(1 \cdot 63 \cdot 64) = 3 \cdot 7 \cdot 2 = \mathbf{42}$
$c > rad(abc)$!!!

$c \leq rad(abc)$?

$1 + 63 = \mathbf{64}, rad(1 \cdot 63 \cdot 64) = 3 \cdot 7 \cdot 2 = \mathbf{42}$
$c > rad(abc)$!!!
Maybe there are only finitely many exceptional cases $a + b = c$?

$c \leq rad(abc)$?

$1 + 63 = \mathbf{64}, rad(1 \cdot 63 \cdot 64) = 3 \cdot 7 \cdot 2 = \mathbf{42}$
$c > rad(abc)$!!!
Maybe there are only finitely many exceptional cases $a + b = c$?
It is still NOT true.

$c \leq rad(abc)$?

$1 + 63 = \mathbf{64}, rad(1 \cdot 63 \cdot 64) = 3 \cdot 7 \cdot 2 = \mathbf{42}$
$c > rad(abc)$!!!
Maybe there are only finitely many exceptional cases $a + b = c$?
It is still NOT true.

Example

$a = 1, b = 2^{6n} - 1, c = 2^{6n}, n > 1$

$c \leq rad(abc)$?

$1+63=\mathbf{64}, rad(1\cdot 63\cdot 64)=3\cdot 7\cdot 2=\mathbf{42}$
$c > rad(abc)$!!!
Maybe there are only finitely many exceptional cases $a+b=c$?
It is still NOT true.

Example

$a=1, b=2^{6n}-1, c=2^{6n}, n>1$
$a+b=1+(2^{6n}-1)=2^{6n}=c$

$c \leq rad(abc)$?

$1 + 63 = \mathbf{64}, rad(1 \cdot 63 \cdot 64) = 3 \cdot 7 \cdot 2 = \mathbf{42}$
$c > rad(abc)$!!!
Maybe there are only finitely many exceptional cases $a + b = c$?
It is still NOT true.

Example

$a = 1, b = 2^{6n} - 1, c = 2^{6n}, n > 1$
$a + b = 1 + (2^{6n} - 1) = 2^{6n} = c$

$c \leq rad(abc)$?

$1 + 63 = \mathbf{64}, rad(1 \cdot 63 \cdot 64) = 3 \cdot 7 \cdot 2 = \mathbf{42}$
$c > rad(abc)$!!!
Maybe there are only finitely many exceptional cases $a + b = c$?
It is still NOT true.

Example

$a = 1, b = 2^{6n} - 1, c = 2^{6n}, n > 1$
$a + b = 1 + (2^{6n} - 1) = 2^{6n} = c$

$c > rad(abc)$ for infinitely many $a + b = c$

$$
\begin{aligned}
rad(abc) &= rad(2(2^{6n} - 1)) \\
&= 2rad(2^{6n} - 1) \\
&= 2rad(9 \cdot \frac{b}{9}) \\
&\leq 2 \cdot 3 \cdot \frac{b}{9} \\
&= \frac{2}{3}b \\
&< \frac{2}{3}c
\end{aligned}
$$

"$rad(abc)$ should not be small compared to c."

How about raising the power of $rad(abc)$?

Conjecture ($abc-$Conjecture, Masser and Oestelé 1988))

For every $\varepsilon > 0$, there are only finite many triple (a, b, c) of coprime positve integers with $a + b = c$ such that:

$$c > rad(abc)^{1+\varepsilon}.$$

How about raising the power of $rad(abc)$?

Conjecture ($abc-$Conjecture, Masser and Oestelé 1988))

For every $\varepsilon > 0$, there are only finite many triple (a, b, c) of coprime positve integers with $a + b = c$ such that:

$$c > rad(abc)^{1+\varepsilon}.$$

Equivalently, we have

Conjecture ($abc-$Conjecture)

For every $\varepsilon > 0$, there exists a constant K_ε such that for all triples (a, b, c) of coprime positve integers, with $a + b = c$ such that

$$c < K_\varepsilon rad(abc)^{1+\varepsilon}.$$

Explicit $abc-$conjecture

Conjecture (Explicit $abc-$conjecture, Baker 2004, Laishram and Shorey 2012 ($\varepsilon = 3/4$))

For all triples (a, b, c) of coprime positive integers, with $a + b = c$ such that

$$c < (rad(abc))^{1+\frac{3}{4}}.$$

Fermat's Last Theorem (by Explicit abc)

Theorem

$x^n + y^n = z^n$ has no integer solutions for $n > 2$ and $xyz \neq 0$.

Proof.

WLOG, assume that $\gcd(x, y, z) = 1$. By the explicit abc-conjecture,

$$z^n < rad(xyz)^{1+3/4} < (z^3)^2 = z^6.$$

Thus, $n \leq 5$.
$n = 4$ by Fermat 1636
$n = 3$ by Euler 1770
$n = 5$ by Dirichlet 1825 □

abc Conjecture for Function Fields

Theorem (Mason-Stothers Theorem, 1981)

Let K be a field of characteristic 0. If $a(t), b(t), c(t)$ *are nonzero polynomials in* $K[t]$ *with* $a(t) + b(t) = c(t)$ *and* $\gcd(a(t), b(t), c(t)) = 1$*, then*

$$\max\{\deg a, \deg b, \deg c\} \leq rad(abc) - 1.$$

Example

$a(t) = 1, b(t) = t^n, c(t) = t^n + 1$

abc Conjecture for Function Fields

Theorem (Mason-Stothers Theorem, 1981)

Let K be a field of characteristic 0. If $a(t), b(t), c(t)$ are nonzero polynomials in $K[t]$ with $a(t) + b(t) = c(t)$ and $\gcd(a(t), b(t), c(t)) = 1$, then

$$\max\{\deg a, \deg b, \deg c\} \leq rad(abc) - 1.$$

Example

$a(t) = 1, b(t) = t^n, c(t) = t^n + 1$
$\max\{\deg a, \deg b, \deg c\} = n$

abc Conjecture for Function Fields

Theorem (Mason-Stothers Theorem, 1981)

Let K be a field of characteristic 0. If $a(t), b(t), c(t)$ are nonzero polynomials in $K[t]$ with $a(t) + b(t) = c(t)$ and $\gcd(a(t), b(t), c(t)) = 1$, then

$$\max\{\deg a, \deg b, \deg c\} \leq rad(abc) - 1.$$

Example

$a(t) = 1, b(t) = t^n, c(t) = t^n + 1$
$\max\{\deg a, \deg b, \deg c\} = n$
$rad(abc) = t(t^n + 1),$

Theorem (Mason-Stothers Theorem, 1981)

Let K be a field of characteristic 0. If $a(t), b(t), c(t)$ are nonzero polynomials in $K[t]$ with $a(t) + b(t) = c(t)$ and $\gcd(a(t), b(t), c(t)) = 1$, then

$$\max\{\deg a, \deg b, \deg c\} \leq rad(abc) - 1.$$

Example

$a(t) = 1, b(t) = t^n, c(t) = t^n + 1$
$\max\{\deg a, \deg b, \deg c\} = n$
$rad(abc) = t(t^n + 1), \deg rad(abc) = n + 1$

Theorem

There are no nonzero polynomials $x(t), y(t), z(t) \in K[t]$ *such that*

$$x(t)^n + y(t)^n = z(t)^n, \ \textit{for } n > 2.$$

Fermat's Last Theorem for Polynomials

Theorem

There are no nonzero polynomials $x(t), y(t), z(t) \in K[t]$ *such that*

$$x(t)^n + y(t)^n = z(t)^n, \text{ for } n > 2.$$

Proof.

Exercise □

How about $f_c(x) = x^d + c$ over $\mathbb{Q}$?

- For every $d \in \mathbb{N}$, there are infinitely many $c \in \mathbb{Q}$ such that $f_c(x) = x^d + c$ has a fixed point in $\mathbb{Q}$, i.e., $c = t - t^d, t \in \mathbb{Q}$.

How about $f_c(x) = x^d + c$ over $\mathbb{Q}$?

- For every $d \in \mathbb{N}$, there are infinitely many $c \in \mathbb{Q}$ such that $f_c(x) = x^d + c$ has a fixed point in $\mathbb{Q}$, i.e., $c = t - t^d, t \in \mathbb{Q}$.
- For every odd $d \geq 3$ and $n \geq 2$, $f_c(x)$ has no n-periodic point in $\mathbb{Q}$.

How about $f_c(x) = x^d + c$ over $\mathbb{Q}$?

- For every $d \in \mathbb{N}$, there are infinitely many $c \in \mathbb{Q}$ such that $f_c(x) = x^d + c$ has a fixed point in $\mathbb{Q}$, i.e., $c = t - t^d, t \in \mathbb{Q}$.
- For every odd $d \geq 3$ and $n \geq 2$, $f_c(x)$ has no n-periodic point in $\mathbb{Q}$.
- For every even $d \geq 2$, $f_{-1}(x) = x^d - 1$ has "trivial" 2-periodic points: $1 \longrightarrow 0 \longleftrightarrow -1$.

How about $f_c(x) = x^d + c$ over $\mathbb{Q}$?

- For every $d \in \mathbb{N}$, there are infinitely many $c \in \mathbb{Q}$ such that $f_c(x) = x^d + c$ has a fixed point in $\mathbb{Q}$, i.e., $c = t - t^d, t \in \mathbb{Q}$.
- For every odd $d \geq 3$ and $n \geq 2$, $f_c(x)$ has no n-periodic point in $\mathbb{Q}$.
- For every even $d \geq 2$, $f_{-1}(x) = x^d - 1$ has "trivial" 2-periodic points: $1 \longrightarrow 0 \longleftrightarrow -1$.
- For d=4 and $n = 2$, there are infinitely many $c \in \mathbb{Q}$ such that $f_c(x) = x^4 + c$ has 2-periodic points.

$$F_2^*(x,c) = \frac{f_c^{(2)}(x) - x}{f_c(x) - x} = 0 \rightsquigarrow E : y^2 = x^3 - 4$$

Ex: $f_c(x) = x^4 - \frac{19561}{10000}, \frac{9}{10} \longleftrightarrow \frac{-13}{10}$

How about $f_c(x) = x^d + c$ over $\mathbb{Q}$?

Questions?

1. What about 2-periodic points of $f_c(x) = x^6 + c$?

How about $f_c(x) = x^d + c$ over $\mathbb{Q}$?

Questions?

1. What about 2-periodic points of $f_c(x) = x^6 + c$?

$$F_2^*(x,c) = \frac{f_c^{(2)}(x) - x}{f_c(x) - x} = 0 \rightsquigarrow H : y^2 = x^5 + 81$$

How about $f_c(x) = x^d + c$ over $\mathbb{Q}$?

Questions?

1. What about 2-periodic points of $f_c(x) = x^6 + c$?

   $$F_2^*(x,c) = \frac{f_c^{(2)}(x) - x}{f_c(x) - x} = 0 \rightsquigarrow H : y^2 = x^5 + 81$$

2. What about 2-periodic points of $f_c(x) = x^d + c$, for $d = 8, 10, 12, \ldots$?

How about $f_c(x) = x^d + c$ over $\mathbb{Q}$?

Questions?

1. What about 2-periodic points of $f_c(x) = x^6 + c$?
   $$F_2^*(x, c) = \frac{f_c^{(2)}(x) - x}{f_c(x) - x} = 0 \rightsquigarrow H : y^2 = x^5 + 81$$
2. What about 2-periodic points of $f_c(x) = x^d + c$, for $d = 8, 10, 12, \ldots$?
3. Is it true that $f_c(x) = x^d + c \in \mathbb{Q}(x)$ does not have rational 3-periodic points for $d \geq 3$?

How about $f_c(x) = x^d + c$ over $\mathbb{Q}$?

Questions?

1. What about 2-periodic points of $f_c(x) = x^6 + c$?
$$F_2^*(x,c) = \frac{f_c^{(2)}(x) - x}{f_c(x) - x} = 0 \rightsquigarrow H : y^2 = x^5 + 81$$
2. What about 2-periodic points of $f_c(x) = x^d + c$, for $d = 8, 10, 12, \ldots$?
3. Is it true that $f_c(x) = x^d + c \in \mathbb{Q}(x)$ does not have rational 3-periodic points for $d \geq 3$?
4. Is it true that $f_c(x) = x^d + c \in \mathbb{Q}(x)$ does not have rational n-periodic points for $d \geq 3$ and $n \geq 3$?

$f_{d,c}(x) = x^d + c$

Theorem (Narkierwicz, 2013)

For $n > 1$ and $d > 2$ odd, there is no $c \in \mathbb{Q}$ such that $f_{d,c}$ has a $\mathbb{Q}$-rational periodic point of minimal period n. Furthermore,

$$\#PrePer(f_{d,c}, \mathbb{Q}) \leq 4.$$

Conjecture (Hutz, 2015)

For $n > 2$ there is no even $d > 2$ and $c \in \mathbb{Q}$ such that $f_{d,c}$ has a $\mathbb{Q}$-rational periodic point of minimal period n. Furthermore,

$$\#PrePer(f_{d,c}, \mathbb{Q}) \leq 4.$$

2-Periodic Points: Dynatomic Polynomial Method

Theorem

There are infinitely many $c \in \mathbb{Q}$ such that $f_c(z) = z^4 + c$ has rational periodic points of exact period 2.

Theorem (Bremner, Hindry, Merél, P., Waldschmidt 2019+ (BHMPW))

1. *Let* $H_{81} : y^2 = x^5 + 81$. *Then* $H_{81}(\mathbb{Q}) = \{(2, \pm 7), (0, \pm 9), (, 3, \pm 18), \infty\}$
2. *Let* $f_c(z) = z^6 + c \in \mathbb{Q}[z]$. *If* $f_c(x)$ *has rational point of period 2, then* $c = -1$.

Note $rank(H_{81}) = 2$.

Benedetto's Trick

Period n of $f_c = x^{2k} + c$: $f_c(x_i) = x_{i+1}, f(x_n) = x_1$

$$\prod_{i=1}^{n}(x_i - x_{i+1}) = \prod_{i=1}^{n}((x_i^{2k} + c) - (x_{i+1}^{2k} + c))$$

$$= \prod_{i=1}^{n}((x_i^k - x_{i+1}^k)(x_i^k + x_{i+1}^k))$$

$$1 = \prod_{i=1}^{n}\left(\frac{(x_i^k - x_{i+1}^k)}{x_i - x_{i+1}}(x_i^k + x_{i+1}^k)\right)$$

For period $n = 2$,

$$z^{2k-1} = x^k + y^k$$

where $x_1 = x/z, x_2 = y/z$ and $\gcd(x,z) = \gcd(y,z) = \gcd(x,y) = 1$.

2-Periodic Points: Modular Method

Theorem (Darmon, Merel 1997)

Let the exponent n be an arbitrary positive integer.

1. *The equation $x^n + y^n = 2z^n$ has no nontrivial primitive solution when $n \geq 3$.*
2. *The equation $x^n + y^n = z^2$ has no nontrivial primitive solution when $n \geq 4$.*
3. *The equation $x^n + y^n = z^3$ has no nontrivial primitive solution when $n \geq 3$.*

Theorem (Ellenberg 2004, Bennett, Ellenberg, Ng 2010)

There are no solutions in coprime integers x, y, z to the equation $x^4 + y^2 = z^n$ with $\gcd(x,y,z) = 1$ and $n \geq 4$. The only solution in positive coprime integers x, y, z to the equation $x^4 + 2y^2 = z^n$ with $n \geq 4$ is $(x,y,z,n) = (1,11,3,5)$.

2-Periodic Points: Modular Method

Theorem (Freitas 2016)

Let $p \geq 17$ be a prime satisfying $(-3/p) = -1$, that is $p \equiv 2 \mod 3$. Then equation $x^3 + y^3 = z^p$ has no non-trivial primitive solutions.

Theorem (BHMPW)

$f_c(x) = x^{2k} + c \in \mathbb{Q}$ *has no non-trivial 2-periodic points when*

1. $k \equiv 0, 2 \mod 3$
2. $k = 4l$ *for some* $l \in \mathbb{N}$

$n-$Periodic Points: Easy as abc

Theorem (BHMPW)

Let n be an integer greater than 1 and $f_c(x) = x^d + c \in \mathbb{Q}[x]$ where $d > 6$. If the abc$-$conjecture (explicit form) is valid, then f_c has no non-trivial rational periodic points of exact period n.

Easy as abc: Number Fields

Conjecture (abc Conjecture: Number Fields, Elkies 1991)

For any given $\varepsilon > 0$*, if* $a+b+c=0$*, where* a, b*, and* c *are algebraic numbers in some number field* K *then*

$$H(a,b,c) \ll_{\varepsilon} (\Delta_K N(a,b,c))^{1+\varepsilon}.$$

Theorem (BHMPW)

Let n *be an integer greater than* 1 *and* $f_c(x) = x^d + c \in K[x]$*,* $c \notin \mathscr{O}_K$*. If the* $abc-$*conjecture is valid, then there is* $C_K \in \mathbb{N}$ *such that* f_c *has no rational periodic points of exact period* n *for all* $d \geq C_K$*. The constant* C_K *depends only on* Δ_K*.*

Figure: ABC?

THANK YOU FOR YOUR ATTENTION!