Applying Graph Theory to Problems in Traffic Control

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Königsberg Bridge Problem

The city of Königsberg (now, Kaliningard, Russia) was located of the Pregel (Pregolya) River.

“Does there exist a walk crossing each of the seven bridges of Königsberg exactly once?”

Leonhard Euler: In 1736, the beginning of graph theory.
Königsberg Bridge Problem

\[ G = (V, E) : V = \{u, v, x, y\} \quad \text{and} \quad E = \{1, 2, 3, 4, 5, 6, 7\} \]

"A connected graph has an Eulerian trail if and only if exactly zero or two vertices have odd degree."
The figure below shows the famous maze at *Hampton Court* in England. The essential features of this maze are passages of pathways, separated by hedges and merged at a certain number of junctions.

**Can you escape this maze by using graph?**

**How many ways can you escape from this maze?**
Phasing of Traffic Lights

Graph Theory in Traffic Control

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**Phasing of Traffic Lights**

**Intersection Graphs**

- $\mathcal{F} = \{S_1, S_2, \ldots, S_m\}$ (a family of sets)
- The vertices are the sets in $\mathcal{F}$.
- There is an edge between two sets $S_i$ and $S_j$ if and only if they have a nonempty intersection.

**Example**

Let $S_1 = \{a, b, c, d\}$, $S_2 = \{c, e, h\}$, $S_3 = \{e, f, g\}$, $S_4 = \{h\}$.

\begin{itemize}
  \item $S_1 = \{a, b, c, d\}$
  \item $S_2 = \{c, e, h\}$
  \item $S_3 = \{e, f, g\}$
  \item $S_4 = \{h\}$
\end{itemize}
Theorem. (Marczewski, 1954)

Every graph is isomorphic to the intersection graph of some family of sets.

Given $G$, let $S(u) = \{\{u, v\} : \{u, v\} \in E(G)\} \cup \{u\}$.

Then for all $u \neq v$ in $V(G)$, $\{u, v\} \in E(G)$ iff $S(u) \cap S(v) \neq \emptyset$. 
Phasing of Traffic Lights

Interval Graphs

The intersection graph of a family of intervals on the real line.

Example

Let $J(a) = (1, 5)$, $J(b) = (4, 8)$, $J(c) = (1, 14)$, $J(d) = (7, 11)$, and $J(e) = (10, 14)$.
Let $G$ be a graph. Is it isomorphic to an interval graph?

Is a cycle of length 3, $C_3$, an interval graph?

Is $C_{n \geq 4}$ an interval graph?
Phasing of Traffic Lights

- \( H = (W, F) \) is a **subgraph** of \( G = (V, E) \) if \( W \subseteq V \) and \( F \subseteq E \).
- \( H \) is a **generated subgraph** if \( F \) consists of all edges from \( E \) joining vertices in \( W \). (Induced subgraph)

If \( G \) is an interval graph, ...

⇒ every generated subgraph must be an interval graph.
⇒ it has the property that no graph \( C_{n \geq 4} \) is a generated subgraph.

(Lekkerkerker and Boland, 1962)

⇔ it satisfies the above property and has no asteroidal triple.
Circular arc graphs

An intersection graph of arcs on a given circle.

- Every interval graph is a circular arc graph, but the converse is false, see $C_4$. 
There are various traffic streams approaching the traffic intersection.

**Figure: Samranrat Junction**

Two traffic streams are **compatible** with each other if they can be moving at the same time without dangerous consequences.
Phasing of Traffic Lights

Compatibility Graph

- The vertices are the traffic streams,
- Two streams are joined by an edge if and only if they are compatible.
Phasing of Traffic Lights

A cycle of green and red lights $\Rightarrow$ a clock circle.

A feasible green light assignment $= \{ \text{arcs of the circle} \}$

Compatible traffic streams are allowed to receive overlapping arcs.
Phasing of Traffic Lights

A cycle of green and red lights \( \Rightarrow \) a clock circle.

A feasible green light assignment = \{ arcs of the circle \}

The corresponding intersection graph of arcs on the circle (a circular arc graph)
$H$ is a **spanning subgraph** of $G$.

$G$ is not an interval graph, but $H$ is.
Phasing of Traffic Lights

What makes one feasible green light assignment better than another?
Phasing of Traffic Lights

For example,

- Minimize the total amount of waiting time
  ⇒ Minimize the total amount of red light times in a cycle.

- Information about expected arrival times of different traffic streams
  ⇒ Penalize starting times for being far from the traffic stream’s expected arrival time
  ⇒ Minimize the penalties.
Phasing of Traffic Lights

Minimize the total red light times (Stoffers, 1968)

- Each green light arc must be a certain minimal length.
- Generate corresponding feasible green light assignments by considering orderings of maximal cliques, which are consecutive, $K_1, K_2, \ldots, K_m$.

A consecutive ordering of maximal cliques:
$K_1 = \{e, b\}$,
$K_2 = \{b, a, d\}$,
$K_3 = \{d, c\}$,
$K_4 = \{c, f\}$
There are 4 phases. Assign a duration to each phase (clique):

- \( K_1 = \{e, b\} \), \( \Rightarrow d_1 \),
- \( K_2 = \{b, a, d\} \), \( \Rightarrow d_2 \),
- \( K_3 = \{d, c\} \), \( \Rightarrow d_3 \),
- \( K_4 = \{c, f\} \), \( \Rightarrow d_4 \)

What should the duration \( d_i \) be so that the sum of the red light times is as small as possible?
Phasing of Traffic Lights

There are 4 phases.

- \( K_1 = \{e, b\} \),
- \( K_2 = \{b, a, d\} \),
- \( K_3 = \{d, c\} \),
- \( K_4 = \{c, f\} \)

Assign a duration to each phase (clique):

- \( \Rightarrow d_1 \),
- \( \Rightarrow d_2 \),
- \( \Rightarrow d_3 \),
- \( \Rightarrow d_4 \)

- \( a \Rightarrow \text{Red light: } K_1, K_3, K_4 \Rightarrow \text{Total red light} = d_1 + d_3 + d_4 \)
- \( b \Rightarrow \text{Red light: } K_3, K_4 \Rightarrow \text{Total red light} = d_3 + d_4 \)
- \( c \Rightarrow \text{Red light: } K_1, K_2 \Rightarrow \text{Total red light} = d_1 + d_2 \)
- \( d \Rightarrow \text{Red light: } K_1, K_4 \Rightarrow \text{Total red light} = d_1 + d_4 \)
- \( e \Rightarrow \text{Red light: } K_2, K_3, K_4 \Rightarrow \text{Total red light} = d_2 + d_3 + d_4 \)
- \( f \Rightarrow \text{Red light: } K_1, K_2, K_3 \Rightarrow \text{Total red light} = d_1 + d_2 + d_3 \)
The total red light time for all traffic streams

\[(d_1 + d_3 + d_4) + (d_3 + d_4) + (d_1 + d_2) + (d_1 + d_4) + (d_2 + d_3 + d_4) + (d_1 + d_2 + d_3).\]

- The minimum green light time for a stream = 20 seconds.
- The total cycle = 120 seconds

\[
\min 4d_1 + 3d_2 + 4d_3 + 4d_4, \quad d_i \geq 0
\]

\[
d_2 \geq 20, \quad \text{(Steam a)}
\]
\[
d_1 + d_2 \geq 20, \quad \text{(Steam b)}
\]
\[
d_3 + d_4 \geq 20, \quad \text{(Steam c)}
\]
\[
d_2 + d_3 \geq 20, \quad \text{(Steam d)}
\]
\[
d_1 \geq 20, \quad \text{(Steam e)}
\]
\[
d_4 \geq 20, \quad \text{(Steam f)}
\]

\[d_1 + d_2 + d_3 + d_4 = 120.\]
Phasing of Traffic Lights

\[
\min 4d_1 + 3d_2 + 4d_3 + 4d_4, \quad d_i \geq 0
\]

\[
d_2 \geq 20, \quad \text{(Steam a)} \quad (1)
\]
\[
d_1 + d_2 \geq 20, \quad \text{(Steam b)} \quad (2)
\]
\[
d_3 + d_4 \geq 20, \quad \text{(Steam c)} \quad (3)
\]
\[
d_2 + d_3 \geq 20, \quad \text{(Steam d)} \quad (4)
\]
\[
d_1 \geq 20, \quad \text{(Steam e)} \quad (5)
\]
\[
d_4 \geq 20, \quad \text{(Steam f)} \quad (6)
\]
\[
d_1 + d_2 + d_3 + d_4 = 120. \quad (7)
\]

⇒ Minimize \( d_1 + d_3 + d_4 \).
⇒ From (5) and (6), \( d_1, d_4 \geq 20 \) and \( d_3 \geq 0 \).
⇒ \( d_1 = d_4 = 20, d_3 = 0, d_2 = 80 \).
To find an optimal feasible green light assignment,...

1. Identify each circular arc graph $H$,
2. For each circular arc graph, find all the different consecutive orderings of maximal cliques,
3. For each such ordering, find an optimal solution of phase duration,
4. Put all of them together to find an optimal solution for the entire graph.
One-Way Street Problem

- Making certain streets one-way would move traffic more efficiently.
- Is it possible to make certain designated streets one-way?
- In such a way that it is still possible to get from any place to any other place.
One-Way Street Problem

Every street is two-way.

- The street corners ⇒ the vertices of a graph.
- There is an edge between two vertices if and only if their corresponding street corners are joined by a two-way street.

⇒ Place a direction on each edge of this graph (Orientation) ⇒ Digraph—it is possible to get from any place to any other place.
One-Way Street Problem

**Digraphs**

- **A directed graph** or digraph consists of a finite set $V$ of vertices and a set $A$ of arcs (ordered pairs of vertices), $D = (V, A)$.

- A **path** in $D$ is a sequence $u_1, a_1, u_2, a_2, \ldots, a_n, u_{n+1}$, where $u_i$ is a vertex and $a_i$ is an arc from $u_i$ to $u_{i+1}$.

- A digraph is **strongly connected** if for every pair of vertices $u$ and $v$, there is a path from $u$ to $v$ and a path from $v$ to $u$. 

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![Diagrams](diagrams.png)
Every street is two-way.

⇒ A strongly connected digraph.

Does every graph $G$ have a strongly connected orientation? NO!

$\leftarrow$ Bridge (cut-edge)

Theorem 1. (Robbins, 1939)

A graph $G$ has a strongly connected orientation if and only if $G$ is connected and has no bridges.
One-Way Street Problem

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Does every graph \( G \) have a strongly connected orientation? NO!

\[ \Leftrightarrow \text{Bridge (cut-edge)} \]

**Theorem 1. (Robbins, 1939)**

A graph \( G \) has a strongly connected orientation if and only if \( G \) is connected and has no bridges.
Some streets are two-way.

- A mixed graph $G$ with underlying digraph $D$: a set of vertices (street corners), some joined by (one-way) arcs and some joined by undirected edges (two arcs on two-way streets).
- $G$ is strongly connected if $D$ is strongly connected.
- An undirected edge $e$ in $G$ is a bridge if $G \setminus e$ is not connected.

Strong; Not Strong; Strong
Some streets are two-way.

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- $G$ is strongly connected if $D$ is strongly connected.
- An undirected edge $e$ in $G$ is a bridge if $G \setminus e$ is not connected.
Theorem 2. (Boesch and Tindell, 1977)

Suppose $G$ is a strongly connected mixed graph. Then for every edge $e$ of $G$ which is not a bridge, there is an orientation of $e$ so that the resulting mixed graph is still strongly connected.
Lemma 3.

Suppose $G$ is strongly connected mixed graph and $e = \{u, v\}$ is an edge of $G$. Let $D'$ be the digraph obtained from $D$ by deleting arcs $(u, v)$ and $v, u)$. Let $X$ be the set of all vertices reachable from $u$ by a path in $D'$, less the vertex $u$. Let $Y$ be defined similarly from $v$. Suppose $u$ is not in $Y$ and $v$ is not in $X$. Then the edge $e$ must be a bridge of $G$. 
Theorem 2. (Boesch and Tindell, 1977)
Suppose $G$ is a strongly connected mixed graph. Then for every edge $e$ of $G$ which is not a bridge, there is an orientation of $e$ so that the resulting mixed graph is still strongly connected.

Theorem 1. (Robbins, 1939)
A graph $G$ has a strongly connected orientation if and only if $G$ is connected and has no bridges.

The converse can be proved by orienting one edge at a time.
One-Way Street Problem

How can we find a one-way street assignment?

Depth first search in a connected, bridgeless graph (Robert, 1976):

Lower number $\rightarrow$ higher number in the labeling procedure, and otherwise higher number $\rightarrow$ lower number
One-Way Street Problem

Efficiency

Find orientations which make it possible to get from one place to another with short detours.

- \( d_G(u, v) \): the distance between \( u \) and \( v \) of a connected graph \( G \) (the shortest path between \( u \) and \( v \))
- \( d_D(u, v) \): the distance from \( u \) to \( v \) in a strongly connected digraph \( D \) (the shortest path from \( u \) to \( v \))
- \( d_D(u, v) \) may not be the same as \( d_D(v, u) \).

\[
d_D(u, v) = 11, \quad d_{D'}(u, v) = 3
\]
One-Way Street Problem

1) Find that strongly connected orientation $D$ of $G$ in which the average distance $d_D(u, v)$ over all $u, v$ is as small as possible.

2) Find that strongly connected orientation $D$ of $G$ in which the maximum distance $d_D(u, v)$ over all $u, v$ is minimized.

3) Find that strongly connected orientation $D$ of $G$ in which the difference between the distances $d_G(u, v)$ and $d_D(u, v)$ is on the average as small as possible.

4) Find that strongly connected orientation $D$ of $G$ in which the maximum of the differences between the distances $d_G(u, v)$ and $d_D(u, v)$ is as small as possible.

Inefficiency

Make streets one way so that long distances must be traveled to get from place to place.
References

1. STOFFERS, K. E. 1968. Scheduling of traffic lights - A new approach, Transportation Research, 2, pp. 199-234.


Thank You!