It is Impossible to Solve Chomp(?)

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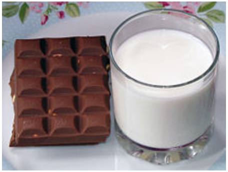
How to Play

David Gale's famous game of Chomp is a two-player strategy game played on a rectangular chocolate bar made up of smaller square blocks. The players take it in turns to choose one block and "eat it" (remove from the board), together with those that are below it and to its right. The top left block is "poisoned" and the player who eats this loses.



credit: Wikipedia

Example

Consider 5-by-3 chocolate bar:

X	X	X
X	X	X
X	X	X
X	X	X
X	X	X

If the first player choose to play (5,3):

X	X	X
X	X	X
X	X	X
X	X	X
X	X	

Then the second player may choose to play (3,2):

X	X	X
X	X	X
X		
X		
X		

Other Take-Away Games

Chomp is one of the game in the category of Take-Away Game. Players take turn to remove "chip" from the pile(s). The last player who makes a move will either win or lose depending on the rule of that game.

Example: Let's play a game of one pile with 42 chips where both players have choices of removing 1 chip or 2 chips or 3 chips from the pile during their turn. The player who removes the last chip wins the game. Is this the winning position for the first player?

Questions to be Answered

Questions:

- Categorize the winning positions.
- For each winning position, give the winning move(s).

Back to Chomp

We will consider all the positions that arise from the m-by-n rectangular chocolate bar.

Notation: position $:= (a_1, a_2, a_3, \ldots, a_n)$ where a_i is the number of squares in row i. i.e. (1,3,4) is the position

X	X	X	X
X	X	X	
X			

.

Some Losing/Winning Positions

The known losing positions are of the form $(1\$k, k+1)$ i.e. $k = 3$

X X X X
X
X
X

On the other hand, it is well known that all the *m*-by-*n* rectangular chocolate bars are the winning positions. (strategy stealing!).

It is not common to be able to tell which position is a winner (winning position) or loser (losing position).

Two Computational Challenges from Doron Zeilberger

First Computational Challenge: Find all the winning moves for a 1001 by 1003 chocolate bar. (a $500 donation to the OEIS in honor of the finder)

Second Computational Challenge: Find a and b such that an a by b Chomp has (at least) THREE winning moves. (a $100 donation to the OEIS in honor of the finder)
Note: Some starting positions have more than one winning move ie. 8 by 10 chocolate bar, 6 by 13 bar and 10 by 14 chocolate bar.

"IN ORDER TO UNDERSTAND SOMETHING REALLY DEEPLY, YOU SHOULD PROGRAM IT." , Doron Zeilberger

"The purpose of computing is insight, not numbers", Richard Hamming

One-Rowed Chomp and Two-Rowed Chomp

We will answer two main questions for One-Rowed Chomp and Two-Rowed Chomp.

One-Rowed Chomp Trivial! The only loser (losing position) is [1].

Two-Rowed Chomp My slow computer program gives the losers as: $(0,1),(1,2),(2,3),(3,4),(4,5),\ldots$.
(The top row has one more square than the bottom row).

Once the losers' positions is conjectured, it could be proved using induction.

Algorithmic Way for Losers of Two-Rowed Chomp

New notation: Given the original notation $(a_1, a_2, \ldots, a_n)$, let $b_1 = a_1$ and $b_i = a_i - a_{i-1}$ and use the square bracket notation, [ ], for the same position. For example, $(1, 3, 4) = [1, 2, 1]$.
Note: In this new notation, the losers of two-rowed Chomp are $[0, 1], [1, 1], [2, 1], [3, 1], [4, 1], \ldots$.

Lemma (Implied Winners Lemma)

If $[a, b]$ is a loser, and $b > 0$ then

$$[a, b + x], \quad 1 \leq x < \infty, \tag{1}$$

$$[a + x, b - x], \quad 1 \leq x \leq b, \tag{2}$$

are all winners. If $[a, 0]$ is a loser then

$$[a + x, y], \quad 0 \leq x, y < \infty, \quad x + y > 0 \tag{3}$$

Algorithmic Way for Losers of Two-Rowed Chomp

Equation (1) say that for each bottom row a, there is at most one b such that $[a, b]$ is a loser.

Let's denote this b by L_a (if exists) and consider the sequence $\{L_a\}_{a=0}^{\infty}$. Fix an integer a, and suppose that we already know L_i for $i < a$, i.e. $[i, L_i]$ are losers for $0 \leq i < a$. Then by equation (2),

$$[a, L_i - (a - i)], \quad 0 \leq i < a$$

are implied winners. This implies that L_a is the smallest non-negative integer that is *not* in the set

$$\{L_{a-1} - 1, L_{a-2} - 2, \ldots, L_0 - a\}.$$

It is now easy to see that $\{L_a\}_{a=0}^{\infty} = 1^{\infty}$.

Trivial to Impossibility and Not-so-much in between

Half of the success in doing research is to find the right problem. If it is too easy, your paper will be rejected. If it is too hard, you might not even do the problem.

- The coefficient of x^k in the expansion $(1+x)^n$ is $\binom{n}{k}$. But there is no closed form formula for the coefficient of x^n in the expansion $(1+x+x^2)^n$.
- $a(n) = 2a(n-1)$ where $a(0) = a_0$ is trivial. But the following is still impossible to understand.

$$a(n+1) = \begin{cases} \frac{a(n)}{2}, & \textit{if} \quad \text{a(n)} \quad \textit{is} \quad \textit{even}; \\ \frac{3a(n)+1}{2}, & \textit{if} \quad \text{a(n)} \quad \textit{is} \quad \textit{odd}. \end{cases}$$

,subject to the *initial condition* $a(0) = a_0$.

Three-Rowed Chomp is Already Very Hard

The losers in Three Rowed Chomp are:

$$[0, a, 1], [1, 0, 2], [1, 1, 0], [2, 0, 2], [2, 1, 2], [2, 2, 2], [2, 3, 2], \ldots$$

In a better notation, assume the all losers are in the form $[c, a, b]$. By the

similar rule to equation (1), there is at most one b such that $[c, a, b]$ is a losing position. Define $B_c(a), \quad a \geq 0$ to be that b (if exists). Then we see that

$$B_0 = 1^\infty,$$

$$B_1 = [2, 0],$$

$$B_2 = 2^\infty,$$

$$B_3 = [3, 3, 0],$$

Three-Rowed Chomp is Already Very Hard

$$B_4 = [4, 4, 0],$$
$$B_5 = [5, 3, 4^{\infty}],$$
$$B_6 = [5, 5, 5, 0],$$
$$B_7 = [6, 6, 3, 5^{\infty}],$$
$$B_8 = [7, 5, 6, 6, 0],$$
$$\ldots$$

Ultimately-periodic

Xinyu Sun (2002) conjectured that, for each c, the set of losing position B_c is either finite or ultimately-periodic.

It was proved by, at-the-time still a high-schooler, Steven Byrnes(2003). It was reexplained again by Doron Zeilberger (2003) which is the main source of material for this talk.

The proof was really nice. I encourage the audience to read the paper and write some programs along (best way to learn new material!).

Final Remark

At the moment, this is what we are at for Chomp . It is very difficult to find the pattern of the losers in three-rowed Chomp. This is due to the nature of the strange recurrence relation of this game!

References

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