CIRCULANT MATRICES

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PRELIMINARY

Denote by A^* the conjugate transpose of A (that is, $A^* = \bar{A}^T$).

- A matrix M is **Hermitian** if $M^* = M$.
- All eigenvalues of a Herminatian matrix are real.
- A Hermitian matrix M is **positive semidefinite** if
$$v^* M v \geq 0$$
for all complex vectors v.
    - All eigenvalues are real and ≥ 0.
- A Hermitian matrix M is **positive definite** if
$$v^* M v > 0$$
for all nonzero complex vectors v.
    - All eigenvalues are real and > 0.
- A matrix M is **unitary** if $MM^* = M^*M = I$ (i.e. $M^* = M^{-1}$)

TRACE INEQUALITY

The following result is well known.

Theorem 1. Let A, B be positive semidefinite matrices of the same size. Then

$$\mathrm{tr}(AB) \leq \mathrm{tr}(A)\mathrm{tr}(B).$$

On taking $A = B$, we can deduce:

Theorem 2. Let A be a positive semidefinite matrix. Then

$$\mathrm{tr}(A^k) \leq \mathrm{tr}(A)^k$$

for all positive integers k.

Question. Can we find a class of matrices A for which

$$\mathrm{tr}(A^k) \geq \mathrm{tr}(A)^k$$

for all positive integers k?

For example, consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

In [56]:
```
A = Matrix(cirl([1,2,3]))
print('tr(A) = ',A.trace())
print('k\t tr(A^k)\t tr(A)^k')
for k in range(2,8):
    print('{}\t {}\t\t {}'.format(k,(A**k).trace(),A.trace()**k))
```

```
tr(A) =  6
k        tr(A^k)         tr(A)^k
2        42              36
3        216             216
4        1314            1296
5        7776            7776
6        46710           46656
7        279936          279936
```

More generally, consider

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

In [49]:
```
A = Matrix(cirl([a,b,c]))
k = 2
((A**k).trace() - A.trace()**k).factor() # calculate tr(A^k) - tr(A)^k
```

Out[49]: $2\left(a^{2} - ab - ac + b^{2} - bc + c^{2}\right)$

In [50]:
```
k = 3
((A**k).trace() - A.trace()**k).factor() # calculate tr(A^k) - tr(A)^k
```

Out[50]: 0

In [51]:
```
k = 4
((A**k).trace() - A.trace()**k).factor() # calculate tr(A^k) - tr(A)^k
```

Out[51]: $2\left(a^{2} - ab - ac + b^{2} - bc + c^{2}\right)^{2}$

In [52]:
```
k = 6
((A**k).trace() - A.trace()**k).factor() # calculate tr(A^k) - tr(A)^k
```

Out[52]: $$2\left(a^2 - ab - ac + b^2 - bc + c^2\right)^3$$

In [53]:
```
k = 8
((A**k).trace() - A.trace()**k).factor() # calculate tr(A^k) - tr(A)^k
```

Out[53]: $$2\left(a^2 - ab - ac + b^2 - bc + c^2\right)^4$$

Note that the expression $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$ for all real a, b, c. This follows from Cauchy-Schwarz Inequality:

$$|u \cdot v| \leq ||u|| \cdot ||v||$$

Taking

$$u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, v = \begin{bmatrix} b \\ c \\ a \end{bmatrix}$$

yields the desired result.

CIRCULANT MATRICES

A **(right) circulant matrix** is a square matrix in which each row is obtained by a right cyclic shift of a row above it.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}, \quad \begin{bmatrix} a0 & a1 & a2 & a3 & a4 \\ a4 & a0 & a1 & a2 & a3 \\ a3 & a4 & a0 & a1 & a2 \\ a2 & a3 & a4 & a0 & a1 \\ a1 & a2 & a3 & a4 & a0 \end{bmatrix}$$

Let's denote by $C(a)$ the circulant matrix whose first row is the row vector a.

$$C([a_0, a_1, a_2, a_3, a_4]) = \begin{bmatrix} a0 & a1 & a2 & a3 & a4 \\ a4 & a0 & a1 & a2 & a3 \\ a3 & a4 & a0 & a1 & a2 \\ a2 & a3 & a4 & a0 & a1 \\ a1 & a2 & a3 & a4 & a0 \end{bmatrix}$$

EIGENVALUES AND EIGENVECTORS

Let a be the row vector $[a_0, a_1, \ldots, a_{n-1}]$. Consider the formal polynomial

$$a(X) = a_0 + a_1 X + a_2 X^2 + \ldots, +a_{n-1} X^{n-1}.$$

Working modulo $X^n - 1$, multiplying it by X shifts the coefficients to right with respect to the order $1, X, X^2, \ldots, X^{n-1}$:

$$Xa(X) = a_{n-1} + a_0 X + a_1 X^2 + a_2 X^3 + \ldots, +a_{n-2} X^{n-1}.$$
$$X^2 a(X) = a_{n-2} + a_{n-1} X + a_0 X^2 + a_1 X + a_2 X^2 + \ldots, +a_{n-3} X^{n-1}.$$

Put it matrix form:

$$\begin{bmatrix} a_0 & a_1 & a_2 & .. & a_{n-1} \\ a_{n-1} & a_0 & a1 & .. & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & .. & a_{n-3} \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \end{bmatrix} \begin{bmatrix} 1 \\ X \\ \vdots \\ X^{n-1} \end{bmatrix} = \begin{bmatrix} a(X) \\ Xa(X) \\ \vdots \\ X^{n-1} a(X) \end{bmatrix} = a(X) \begin{bmatrix} 1 \\ X \\ \vdots \\ X^{n-1} \end{bmatrix}$$

This shows that $C(a)$ has eigenvector

$$v(X) := \begin{bmatrix} 1 \\ X \\ \vdots \\ X^{n-1} \end{bmatrix}$$

with corresponding eigenvalue $a(X) = a_0 + a_1 X + a_2 X^2 + \ldots, +a_{n-1} X^{n-1}$.

Now let $\omega = e^{2\pi i/n}$. For each $X \in \{1, \omega, \ldots, \omega^{n-1}\}$ we get an eigenvector $v(\omega^k)$ with corresponding eigenvalue $a(\omega^k)$. Furthermore, the set

$$\mathcal{B} = \{v(1), v(\omega), \ldots, v(\omega^{n-1})\}$$

forms an **orthogonal basis** for $\mathbb{C}^n$. This is called the **Fourier basis**.

Let F be the "Fourirer Matrix" - the matrix whose columns are normalised Fourier basis. Explicitly,

$$F = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}$$

Then F is unitary and we have

$$F^* C(a) F = D$$

where D is diagonal with diagonal entries $a(1), a(\omega), a(\omega^2), \ldots, a(\omega^{n-1})$

Remark Note that F is independent of the row vector a. Thus all circulant matrices can be diagonalised by the *same* matrix F.

LEFT CIRCULANT MATRICES

A **left circulant matrix** is a square matrix in which each row is obtained by a left cyclic shift of a row above it.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} a0 & a1 & a2 & a3 & a4 \\ a1 & a2 & a3 & a4 & a0 \\ a2 & a3 & a4 & a0 & a1 \\ a3 & a4 & a0 & a1 & a2 \\ a4 & a0 & a1 & a2 & a3 \end{bmatrix}$$

Let's denote by $L(a)$ the left circulant matrix whose first row is the row vector a.

$$L([a_0, a_1, a_2, a_3, a_4]) = \begin{bmatrix} a0 & a1 & a2 & a3 & a4 \\ a1 & a2 & a3 & a4 & a0 \\ a2 & a3 & a4 & a0 & a1 \\ a3 & a4 & a0 & a1 & a2 \\ a4 & a0 & a1 & a2 & a3 \end{bmatrix}$$

RALATIONSHIP WITH CIRCULANT MATRICES

Denote by $R(a)$ the (right) circulant matrix whose first row is the row vector a. Compare $L(a)$ and $R(a)$:

$$L(a) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix} = R(a)$$

We see that

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$$

In general, we have $L(a) = KR(a)$ where K is a block diagonal matrix $\mathrm{diag}([1], \hat{I}\ )$ where $\hat{I}$ is anti diagonal matrix with 1 on the antidiagonal

$$\hat{I} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & .. & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

It can be verified that

$$K^2 = I, \quad \hat{I}^{\,2} = I, \quad F^2 = K, \quad KF = FK$$

EIGENVALUES OF LEFT CIRCULANT MATRICES

Let $L = L(a), R = R(a)$. Recall that R can be diagonalised by the unitary matrix F:

$$F^* RF = D = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}).$$

where

$$\lambda_k = a(\omega^k) = a_0 + a_1\omega^k + a_2\omega^{2k} + \cdots + a_{n-1}\omega^{(n-1)k}$$

are the eigenvalues of R.

Now

$$L = KR = KFDF^* = FKDF^* = F(KD)F^*.$$

This shows that eigenvalues of L are precisely the eigenvalues of KD.

Note the structure of KD:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & .. & 0 \\ 0 & 0 & 0 & 0 & \lambda_{n-1} \end{bmatrix} = \begin{bmatrix} \lambda_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_{n-1} \\ 0 & 0 & 0 & .. & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \end{bmatrix}$$

Thus eigenvalues of L are λ_0 and eigvenvalues of the antidiagonal matrix

$$\text{antidiag}(\lambda_1, \ldots, \lambda_{n-1}) := \begin{bmatrix} 0 & 0 & 0 & \lambda_{n-1} \\ 0 & 0 & .. & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \end{bmatrix}$$

ANTIDIAGONAL MATRICES

Consider

$$A = \text{antidiag}(a_1, \ldots, a_n) = \begin{bmatrix} 0 & 0 & 0 & a_n \\ 0 & 0 & .. & 0 \\ 0 & a_2 & 0 & 0 \\ a_1 & 0 & 0 & 0 \end{bmatrix}$$

With respect to the basis $P = [e_1, e_n, e_2, e_{n-2}, \ldots]$ the matrix A takes the block diagonal form.

$$P^{-1}AP = \begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & .. & 0 \\ 0 & 0 & 0 & B_{n/2} \end{bmatrix} \text{ for even } n$$

$$P^{-1}AP = \begin{bmatrix} B_1 & 0 & 0 & 0 & 0 \\ 0 & .. & 0 & 0 & 0 \\ 0 & 0 & a_{(n+1)/2} & 0 & 0 \\ 0 & 0 & 0 & .. & 0 \\ 0 & 0 & 0 & 0 & B_{(n-1)/2} \end{bmatrix} \text{ for odd } n$$

where

$$B_j = \begin{bmatrix} 0 & a_{n-j+1} \\ a_j & 0 \end{bmatrix}$$

Thus eigenvalues of A are

$$\{\pm\sqrt{a_1 a_n}, \quad \pm\sqrt{a_2 a_{n-1}}, \quad \pm\sqrt{a_3 a_{n-2}} \ldots\} \text{ for even } n$$

or

$$\{a_{(n+1)/2}, \quad \pm\sqrt{a_1 a_n}, \quad \pm\sqrt{a_2 a_{n-1}}, \quad \pm\sqrt{a_3 a_{n-2}} \ldots\} \text{ for odd } n$$

In [169]:

```
A = np.diag([1,2,3,4])
AA = np.flipud(A)
print(AA)
print(Matrix(AA).eigenvals(multiple=True))
```

```
[[0 0 0 4]
 [0 0 3 0]
 [0 2 0 0]
 [1 0 0 0]]
[-2, 2, sqrt(6), -sqrt(6)]
```

In [167]:

```
A = np.diag([1,2,3,4,5])
AA = np.flipud(A)
print(AA)
print(Matrix(AA).eigenvals(multiple=True))
```

```
[[0 0 0 0 5]
 [0 0 0 4 0]
 [0 0 3 0 0]
 [0 2 0 0 0]
 [1 0 0 0 0]]
[3, sqrt(5), -2*sqrt(2), 2*sqrt(2), -sqrt(5)]
```

THE END

THANK YOU!