Measuring Closeness between Cayley Automatic Groups and Automatic Groups

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Automatic groups

Definition

Let G be a finitely generated group. Let $A \subseteq G$ be a finite generating set of the group G. We denote by S the set $S = A \cup A^{-1}$, where A^{-1} is the set of the inverses of elements of A. We say that G is automatic if there exists a regular language $L \subseteq S^*$ such that $\varphi = \pi|_L : L \to G$ is a bijection and for every $a \in A$ the binary relation:

$$R_a = \{(\varphi^{-1}(g), \varphi^{-1}(ga)) | g \in G\} \subseteq L \times L$$

is regular.

References

David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson and William P. Thurston, Word Processing in Groups, 1992, Jones and Barlett Publishers. Boston, MA

Cayley automatic groups

Definition

Let G be a finitely generated group. We say that G is Cayley automatic if for any finite subset $A \subset G$ generating G there exists a bijection $\psi : L \to G$ between a regular language $L \subset \Sigma^*$, where Σ is some finite alphabet, and the group G such that for every $a \in A$ the binary relation:
$L_a = \{(w_1, w_2) \mid w_1, w_2 \in L, \psi(w_1)a = \psi(w_2)\} \subseteq L \times L$ is regular.

- Sénizergues (1991) noticed that a Cayley graph of the Heisenberg group is FA–presentable.
- Khoussainov and Nerode (1994) initiated systematic study of FA–presentable structures.

References

Olga Kharlampovich, Bakhadyr Khoussainov and Alexei Miasnikov, From automatic structures to automatic groups, Groups, Geometry, and Dynamics, V. 8, N 1, 2014, pp. 157–198

Preliminary Definitions

Reformulation of the definition of Cayley automatic groups

Let G be a finitely generated group. Let $A \subseteq G$ be a finite generating set of the group G. We denote by S the set $S = A \cup A^{-1}$, where A^{-1} is the set of the inverses of elements of A. We say that G is Cayley automatic if there exist a regular language $L \subseteq S^*$ and a bijection $\psi : L \to G$ such that for every $a \in A$ the binary relation:
$R_a = \{(\psi^{-1}(g), \psi^{-1}(ga)) | g \in G\} \subseteq L \times L$ is regular.

Proposition(Blumensath)

Every automatically presentable structure has an automatic presentation over the alphabet $\{0, 1\}$.

Preliminary Definitions

Definition

We denote by $\mathfrak{F}$ the following set: $\mathfrak{F} = \{f : [Q, +\infty) \to \mathbb{R}^+ \,|\, [Q, +\infty) \subseteq \mathbb{N} \wedge \forall n (n \in \operatorname{dom} f \implies f(n) \leqslant f(n+1))\}$. Let $f, h \in \mathfrak{F}$. We say that $h \preceq f$ if there exist positive integers K, M and N such that $[N, +\infty) \subseteq \operatorname{dom} h \cap \operatorname{dom} f$ and $h(n) \leqslant Kf(Mn)$ for every integer $n \geqslant N$. We say that $h \asymp f$ if $h \preceq f$ and $f \preceq h$. We say that $h \prec f$ if $h \preceq f$ and $h \not\asymp f$.

Definition

We say that $G \in \mathcal{B}_f$ if there exist a regular language $L \subseteq S^*$ and a Cayley automatic representation $\psi : L \to G$ such that for the function $h \in \mathfrak{F}$, defined by the equation:

$$h(n) = \max\{d_A(\pi(w), \psi(w)) | w \in L^{\leqslant n}\},$$

the inequality $h \preceq f$ holds, where $d_A(g_1, g_2)$ means the distance between g_1 and g_2 in the Cayley graph $\Gamma(G, A)$.

Some first observations and characterization of non–automatic groups

We denote by $\mathcal{A}$ the class of automatic groups and by $\mathcal{C}$ the class of Cayley automatic groups. Let $\mathbf{z} \in \mathfrak{F}$ be the zero function: $\mathbf{z}(n) = 0$, we have: $\mathcal{B}_{\mathbf{z}} = \mathcal{A}$. Let $f, g \in \mathfrak{F}$. If $f \preceq g$, then $\mathcal{A} \subseteq \mathcal{B}_f \subseteq \mathcal{B}_g \subseteq \mathcal{C}$. If $f \asymp g$, then $\mathcal{B}_f = \mathcal{B}_g$.

Proposition

The definition of the class $\mathcal{B}_f$ does not depend on the choice of generating set.

Theorem

Let $d \in \mathfrak{F}$ be any bounded function which is not identically equal to the zero function $\mathbf{z}$. The class $\mathcal{B}_d = \mathcal{A}$. If for any function $f \in \mathfrak{F}$ the class $\mathcal{B}_f$ contains a non–automatic group, then f must be unbounded.

The Baumslag–Solitar groups

Let us consider the Baumslag–Solitar groups $BS(p,q) = \langle a, t | ta^p t^{-1} = a^q \rangle$ with $1 \leqslant p < q$.

Proposition (Normal form)

Any element $g \in BS(p,q)$ for $1 \leqslant p \leqslant q$ can be written uniquely as $g = \widetilde{w}(a,t)a^k$, where $\widetilde{w}(a,t) \in \{t, at, \ldots, a^{q-1}t, t^{-1}, at^{-1}, \ldots, a^{p-1}t^{-1}\}^*$ is freely reduced and $k \in \mathbb{Z}$.

Proposition (Burillo,Elder,2015)

There exist constants $C_1, C_2, D_1, D_2 > 0$ such that for every element $g \in BS(p,q)$ for $1 \leqslant p < q$ written as $\widetilde{w}(a,t)a^k$, we have: $C_1(|\widetilde{w}| + \log(|k|+1)) - D_1 \leqslant d_{\widetilde{A}}(g) \leqslant C_2(|\widetilde{w}| + \log(|k|+1)) + D_2$.

Theorem

Given p and q with $1 \leqslant p < q$, the Baumslag–Solitar group $BS(p,q) \in \mathcal{B}_{\mathfrak{i}}$. Moreover, for any $f \prec \mathfrak{i}$, $BS(p,q) \notin \mathcal{B}_f$.

The lamplighter group

The lamplighter group is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$ of the cyclic group $\mathbb{Z}_2$ and the infinite cyclic group $\mathbb{Z}$. The lamplighter group has the presentation $\langle a, t \mid [t^i a t^{-i}, t^j a t^{-j}], a^2 \rangle$. For a given $i \in \mathbb{Z}$ we put $a_i = t^i a t^{-i}$. The "right–first" normal forms of g is defined as $rf(g) = a_{i_1} a_{i_2} \dots a_{i_k} a_{-j_1} a_{-j_2} \dots\ a_{-j_l} t^m$, where $i_k > \cdots > i_2 > i_1 \geqslant 0$, $j_l > \cdots > j_1 > 0$ and the lamplighter points at the position m.

Proposition (Cleary, Taback,2005)

The word length of the element g with respect to the generating set $A = \{a, t\}$ is given by $d_A(g) = k + l + \min\{2i_k + j_l + |m + j_l|, 2j_l + i_k + |m - i_k|\}$.

Theorem

The lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z} \in \mathcal{B}_{\mathfrak{i}}$. Moreover, for any $f \prec \mathfrak{i}$, $\mathbb{Z}_2 \wr \mathbb{Z} \notin \mathcal{B}_f$.

The Heisenberg group

The Heisenberg group $\mathcal{H}_3(\mathbb{Z})$ is the group of all matrices of the form: $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, where $x, y, z \in \mathbb{Z}$. Every element g of the group $\mathcal{H}_3(\mathbb{Z})$ corresponds to a triple (x, y, z). Let $s = (1, 0, 0)$, $p = (0, 1, 0)$ and $q = (0, 0, 1)$. If $g = (x, y, z)$, then gs, gp and gq are equal to $(x + 1, y, z)$, $(x, y + 1, x + z)$ and $(x, y, z + 1)$, respectively. Let $A = \{e, s, p, q\}$.

Proposition (see, e.g., Roe, Lectures on coarse geometry)

There exist constants C_1 and C_2 such that $C_1(|x| + |y| + \sqrt{|z|}) \leqslant d_A(g) \leqslant C_2(|x| + |y| + \sqrt{|z|})$.

Theorem

The Heisenberg group $\mathcal{H}_3 \in \mathcal{B}_{\mathfrak{e}}$. Moreover, for any $f \prec \sqrt[3]{n}$, $\mathcal{H}_3 \notin \mathcal{B}_f$.

Open Questions

The following questions are apparent from the obtained results:

- Is there any unbounded function $f \prec \mathfrak{i}$ for which the class $\mathcal{B}_f$ contains a non–automatic group?
- Is there any function $f \prec \mathfrak{e}$ for which $\mathcal{H}_3 \in \mathcal{B}_f$?

Is there any way to characterize $\mathcal{B}_{\mathfrak{i}}$ or any other class $\mathcal{B}_f$?

Any Cayley automatic representation for which $n \prec h(n) \prec \exp(n)$?

Any relations with other numerical characteristics of groups?

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