Elliptic integrals and the Jacobi elliptic functions

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We review the definitions, basic properties, and some applications of the Legendre elliptic integrals and the Jacobi elliptic functions. For instance, the perimeter of an ellipse is most simply expressed in terms of the complete Legendre elliptic integral of the second kind, and is why the functions are so named. The Jacobi elliptic functions are functions of two variables, one of which, known as the modulus, k, is normally taken as a parameter. They can be regarded as generalizations of the trigonometric and hyperbolic functions since as $k \to 0$ ($k \to 1$) they tend to the former (latter). A classic example of their application is the expression of the solution of the pendulum equation $\ddot{\theta} + \omega^2 \sin\theta = 0$ in closed form.

Outline

- Perimeter of an ellipse
- Surface area of an ellipsoid
- Properties of elliptic integrals
- Solution of problems in electromagnetism
- Jacobi elliptic functions: snine, cnine, dnine, etc
- Solution of the pendulum equation
- Evaluation of integrals in terms of elliptic functions
- Solution of nonlinear wave equations: cnoidal waves

Perimeter of the ellipse $x^2/a^2 + y^2/b^2 = 1$ *with* $a > b$

Using the parameterization, $x = a\sin\theta$, $y = b\cos\theta$, the arc length from $(x, y) = (0, b)$ to $(a\sin\phi, b\cos\phi)$ is given by

$$\int \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = a\int_0^\phi \sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)\sin^2\theta}\,\mathrm{d}\theta = aE\left(\phi \,\middle|\, 1 - \frac{b^2}{a^2}\right)$$

where $E(\phi|m) = \int_0^\phi \sqrt{1 - m\sin^2\theta}\,\mathrm{d}\theta$ is Legendre's (incomplete) elliptic integral of the 2nd kind. $m \equiv k^2$ where k is called the *modulus*.

Legendre's complete elliptic integral of the 2nd kind, $E(m) \equiv E(\frac{1}{2}\pi|m)$. Thus

$$E(0) = \frac{\pi}{2}, \quad E(1) = 1.$$

The perimeter is $4aE(1 - b^2/a^2)$ which reduces to $2\pi a$ when $m = 0$ (a circle) and $4a$ when $m = 1$ (an infinitely thin ellipse).

Legendre's elliptic integrals (1825)

1st kind (incomplete and complete):

$$F(\phi|m) = \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}\,, \qquad K(m) = F(\tfrac{1}{2}\pi|m)$$

2nd kind (incomplete and complete):

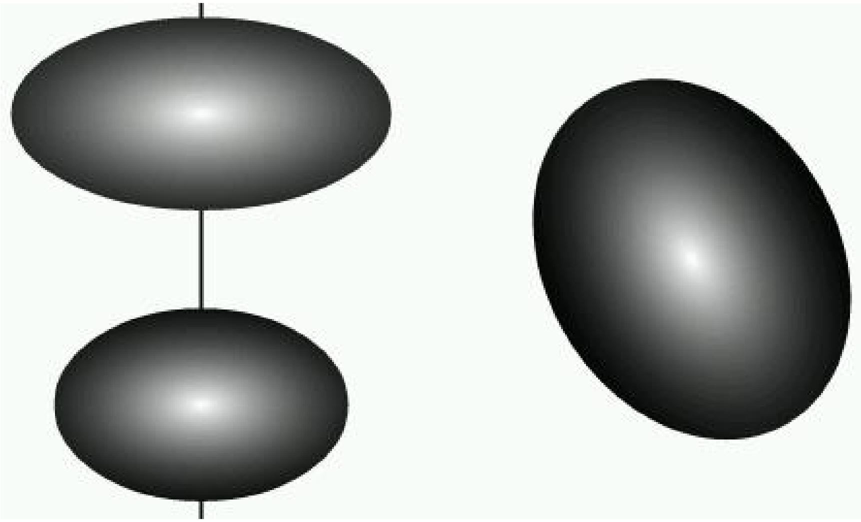
$$E(\phi|m) = \int_0^{\phi} \sqrt{1 - m\sin^2\theta}\,\mathrm{d}\theta, \qquad E(m) = E(\tfrac{1}{2}\pi|m)$$

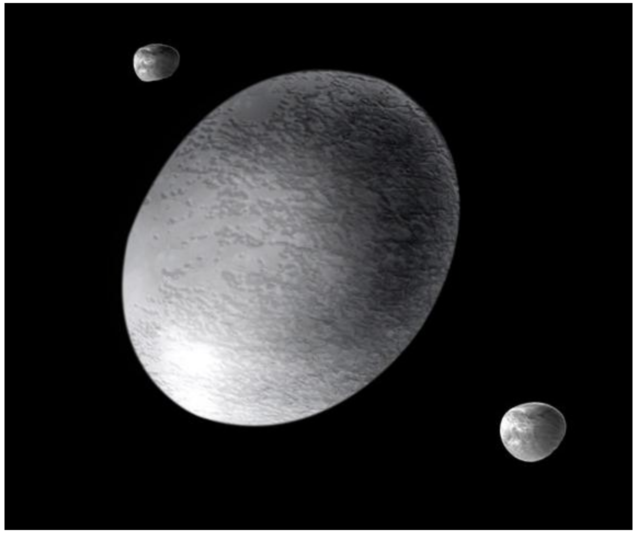
3rd kind (incomplete and complete):

$$\Pi(n;\phi|m) = \int_0^{\phi} \frac{\mathrm{d}\theta}{(1 - n\sin^2\theta)\sqrt{1 - m\sin^2\theta}}\,, \quad \Pi(n|m) = \Pi(n;\tfrac{1}{2}\pi|m)$$

Ellipsoids in nature: Haumea

Discovered in 2004, 35–51 AU from the Sun (Pluto-Sun distance is 30–49 AU), it has 2 moons and a ring system, and is the most rapidly rotating body in the solar system: rotation period is 4 hours; from its light curve it is thought to be a $1920\,\text{km} \times 1540\,\text{km} \times 990\,\text{km}$ ellipsoid.





Minimum energy configurations of a self-gravitating rotating fluid body as angular momentum increases from 0: sphere $\longrightarrow$ Maclaurin (oblate) spheroid $\longrightarrow$ Jacobi ellipsoid $\longrightarrow$ ...

Surface area of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ *with* $a > b > c$

$$2\pi c^2 + \frac{2\pi ab}{\sin\phi}\left(F(\phi|m)\cos^2\phi + E(\phi|m)\sin^2\phi\right)$$

where $\cos\phi = c/a$ and $m = a^2(b^2 - c^2)/[b^2(a^2 - c^2)]$.

The complete integrals: alternative definitions

$$K(m) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}} = \int_0^{\pi/2} \frac{\mathrm{d}\tilde{\theta}}{\sqrt{1 - m\cos^2\tilde{\theta}}}$$

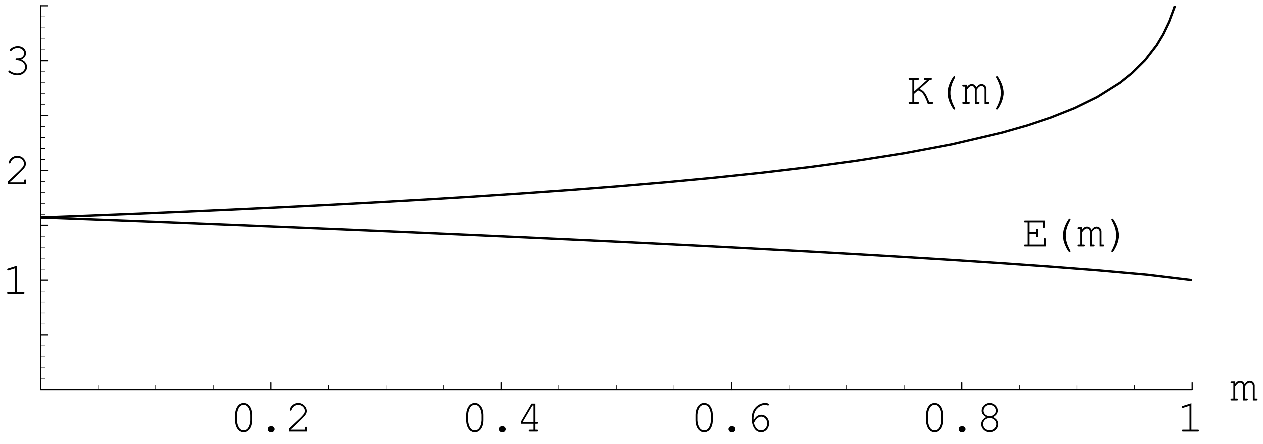
$$= \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-mx^2)}}$$

$$E(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\theta}\ \mathrm{d}\theta = \int_0^{\pi/2} \sqrt{1 - m\cos^2\tilde{\theta}}\ \mathrm{d}\tilde{\theta}$$

$$= \int_0^1 \sqrt{\frac{1-mx^2}{1-x^2}}\,\mathrm{d}x$$

$\tilde{\theta} = \pi/2 - \theta$; $x = \sin\theta$.

The complete integrals: special values and limits



$$K(0) = E(0) = \frac{\pi}{2} \qquad K(m) \sim \log \frac{4}{\sqrt{1-m}} \text{ as } m \to 1- \qquad E(1) = 1$$

The complete integrals: derivatives

$$\frac{\mathrm{d}E(m)}{\mathrm{d}m} = \frac{E(m) - K(m)}{2m}$$

To show this, apply $\partial/\partial b$ to

$$\int_0^{\pi/2} \sqrt{b - m\sin^2\theta}\,\mathrm{d}\theta = \sqrt{b}E(m/b)$$

and set $b = 1$.

$$\frac{\mathrm{d}K(m)}{\mathrm{d}m} = \frac{E(m)}{2(1-m)m} - \frac{K(m)}{2m}$$

Related useful integrals (Good 2001)

For $b > 1$:

$$\int_0^{\pi} \frac{\mathrm{d}\phi}{\sqrt{b \pm \cos\phi}} = \frac{2}{\sqrt{b+1}}\, K\!\left(\frac{2}{b+1}\right)$$

$$\int_0^{\pi} \sqrt{b \pm \cos\phi}\; \mathrm{d}\phi = 2\sqrt{b+1}\, E\!\left(\frac{2}{b+1}\right)$$

($\phi = 2\theta$ and then use $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$.)

... and many more via differentiating under the integral.

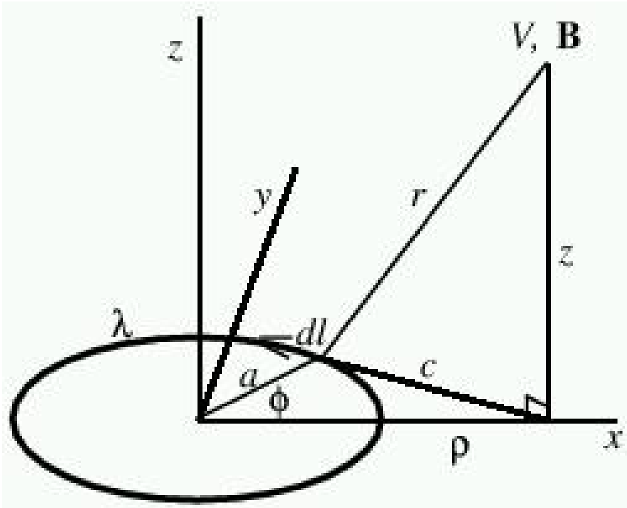
Example: potential from a circular loop of radius a (Good 2001)

Potential at a distance r from a point particle, 'charge' q, with inverse square law: $V(r) = \kappa q/r$. [In electrostatics, q is charge, $\kappa = 1/4\pi\varepsilon_0$.]

For a circular loop at the origin in the plane $z = 0$ with charge per unit length λ,

$$V(\rho, z) = \kappa \oint \frac{\lambda \, \mathrm{d}l}{r}.$$

$$c = \sqrt{a^2 + \rho^2 - 2a\rho\cos\phi}$$



$$V(\rho, z) = 2\kappa \int_0^\pi \frac{\lambda a \, \mathrm{d}\phi}{\sqrt{z^2 + a^2 + \rho^2 - 2a\rho\cos\phi}} = \kappa\lambda\sqrt{\frac{2a}{\rho}} \int_0^\pi \frac{\mathrm{d}\phi}{\sqrt{b - \cos\phi}}$$

$$= \kappa\lambda\sqrt{\frac{8a}{\rho(b+1)}}\, K\left(\frac{2}{b+1}\right), \qquad b = \frac{z^2 + a^2 + \rho^2}{2a\rho}.$$

Inverse elliptic integrals and the Jacobi elliptic function, snine - sn

$$F(\phi|m) = \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}} = \int_0^{u=\sin\phi} \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-my^2)}}$$

Given $F(\phi|m)$, what is ϕ? Answer: $\phi = \mathrm{am}(F(\phi|m)|m)$, the *amplitude* of $F(\phi|m)$. [Thus $\mathrm{am}(u|0) = u$ and $\mathrm{am}(nK(m)|m) = n\pi/2$, $n \in \mathbb{Z}$.]

The Jacobi elliptic function sn (“snine”) is defined as follows:

$$\mathrm{sn}(x|m) := \sin(\mathrm{am}(x|m)) \quad \Rightarrow \quad \sin\phi = \mathrm{sn}(F(\phi|m)|m)$$

$$\Rightarrow \quad u = \mathrm{sn}\left(\int_0^u \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-my^2)}} \,\middle|\, m \right)$$

$$\mathrm{sn}^{-1}(u|m) = \int_0^u \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-my^2)}} \quad \text{cf.} \quad \sin^{-1} u = \int_0^u \frac{\mathrm{d}y}{\sqrt{1-y^2}}$$

and $\mathrm{sn}^{-1}(u|m=1) = \int_0^u \mathrm{d}y/(1-y^2) = \tanh^{-1} u$.

The other Jacobi elliptic functions: cnine - cn*, dnine -* dn*, etc.*

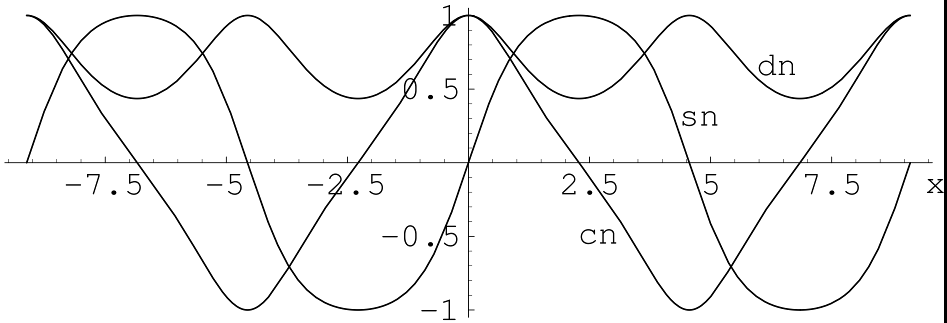
We already have $\mathrm{sn}(x|m) := \sin(\mathrm{am}(x|m))$, so we define its partner similarly:

$$\mathrm{cn}(x|m) := \cos(\mathrm{am}(x|m)) \quad \Rightarrow \quad \mathrm{sn}^2(x|m) + \mathrm{cn}^2(x|m) = 1,$$

$\mathrm{cn}(x|0) = \cos x$, and $\mathrm{cn}(x|1) = \mathrm{sech}\, x$. We also need

$$\mathrm{dn}(x|m) := \sqrt{1 - m\,\mathrm{sn}^2(x|m)} \quad \Rightarrow \quad \mathrm{dn}(x|0) = 1,\ \mathrm{dn}(x|1) = \mathrm{sech}\, x$$

Period of sn, cn is $4K(m)$. They also have imaginary periods of $2\mathrm{i}K'$ and $4\mathrm{i}K'$, respectively, where $K' = K(1-m)$. sn, cn, dn for $m = 0.81$:



$$\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{sn}(x|m) = \mathrm{cn}\,\mathrm{dn}, \quad \frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{cn}(x|m) = -\,\mathrm{sn}\,\mathrm{dn}, \quad \frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{dn}(x|m) = -m\,\mathrm{sn}\,\mathrm{cn}$$

12 functions in total: e.g., $\mathrm{sc}(x|m) \equiv \mathrm{sn}(x|m)/\,\mathrm{cn}(x|m)$, $\mathrm{nd}(x) \equiv 1/\,\mathrm{dn}(x)$.

Solution of the simple pendulum equation $\ddot{\theta} + \omega^2 \sin\theta = 0$

Multiply by $\dot{\theta}$ and integrate to give $\dot{\theta} = \omega\sqrt{2(\cos\theta - \cos\theta_{\mathrm{m}})}$ where the constant of integration has been chosen so that $\dot{\theta} = 0$ when $\theta = \theta_{\mathrm{m}}$, the amplitude.

$$t = \frac{1}{\sqrt{2}\omega}\int^{\theta} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_{\mathrm{m}}}} = \frac{1}{2\omega}\int^{\theta} \frac{\mathrm{d}\theta}{\sqrt{\sin^2 \frac{1}{2}\theta_{\mathrm{m}} - \sin^2 \frac{1}{2}\theta}}.$$

Let $\sin\frac{1}{2}\theta = k\sin\phi$ where $k = \sin\frac{1}{2}\theta_{\mathrm{m}}$. Then

$$t = \frac{1}{\omega}\int^{\sin^{-1}[k^{-1}\sin\frac{1}{2}\theta]} \frac{\mathrm{d}\phi}{\sqrt{1 - k^2\sin\phi}} = \frac{1}{\omega}\,\mathrm{sn}^{-1}(k^{-1}\sin\tfrac{1}{2}\theta|k^2) + t_0$$

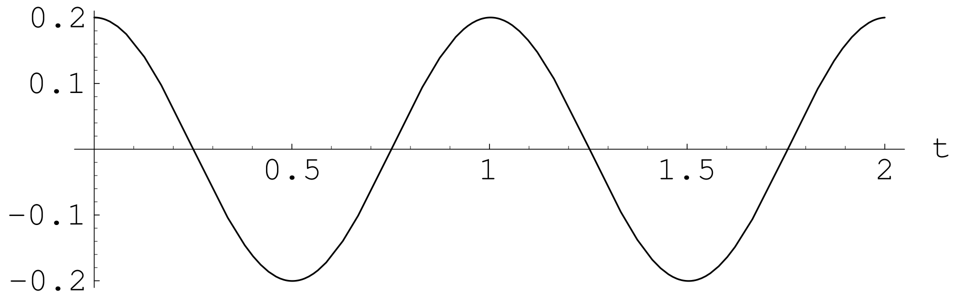
$$\theta = 2\sin^{-1}[k\,\mathrm{sn}(\omega(t - t_0)|k^2)] \quad \text{period } 4K(k^2)/\omega$$

As $\theta_{\mathrm{m}} \to 0$, $k \sim \frac{1}{2}\theta_{\mathrm{m}}$ and $\mathrm{sn} \sim \sin$. Then

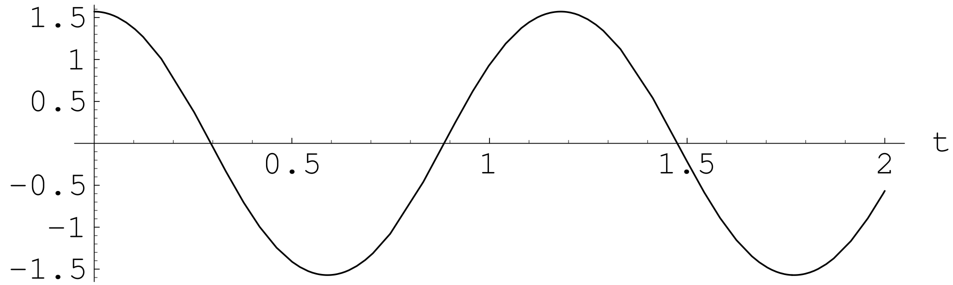
$$\theta = \theta_{\mathrm{m}}\sin\omega(t - t_0) \quad \text{period } 2\pi/\omega$$

Simple pendulum equation solutions ($\omega = 2\pi$)

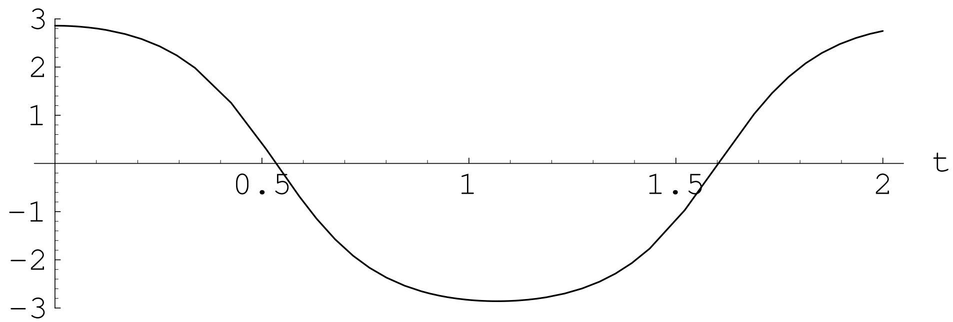
$k = 0.1$



$k = 1/\sqrt{2}$



$k = 0.99$



Evaluation of integrals

Any integral of a rational function of x and $\sqrt{P(x)}$ where $P(x)$ is a cubic or quartic polynomial with distinct roots is known as an *elliptic integral*. Such an integral can be written in terms of Legendre's (3 kinds) of elliptic integrals. These can often be more conveniently expressed in terms of the Jacobi elliptic functions.

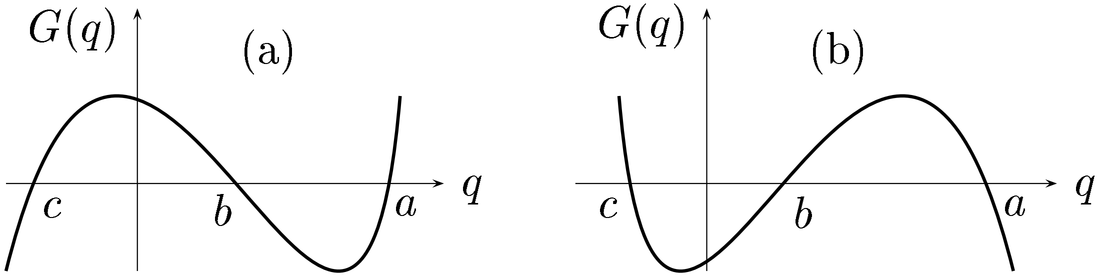
Application: cnoidal waves

Finding a travelling wave solution of a nonlinear PDE sometimes reduces to solving the equation

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \pm\sqrt{G(q)} \quad \Rightarrow \quad z - z_0 = \pm \int^q \frac{\mathrm{d}y}{\sqrt{G(y)}}$$

where z_0 is a constant. If $G(q)$ is a cubic or quartic with distinct roots at least two of which are real, and $G(q)$ is positive between those roots, then there will be a periodic nonlinear wave solution expressible in terms of elliptic functions. Such nonlinear waves are known as *cnoidal waves*.

Cases for having cnoidal waves when $G(q)$ is a cubic



The integrals required to solve cases (a) and (b) are, respectively,

$$\int_c^q \frac{\mathrm{d}y}{\sqrt{(a-y)(b-y)(y-c)}} = \frac{1}{\eta}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{q-c}{b-c}}\,\middle|\,\frac{b-c}{a-c}\right),$$

$$\int_q^a \frac{\mathrm{d}y}{\sqrt{(a-y)(y-b)(y-c)}} = \frac{1}{\eta}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{a-q}{a-b}}\,\middle|\,\frac{a-b}{a-c}\right),$$

where $\eta = \frac{1}{2}\sqrt{a-c}$.

Example: the Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0$$ [subscripts denote differentiation]

Transform to a moving frame $z = x - Vt$ and look for time-independent solutions:

$$(u - V)u_z + u_{zzz} = 0$$

Let $q = u - V$ so $qq_z + q_{zzz}$. Integrate w.r.t. z:

$$\frac{1}{2}q^2 + q_{zz} = \frac{1}{2}B$$

Multiply by $2q_z$ and integrate again:

$$q_z^2 = C + Bq - \frac{1}{3}q^3 = G(q)$$

So for suitable C and B, it is of type (b).

Cnoidal wave solutions of the KdV equation

$$z - z_0 = \mp \int_a^q \frac{\sqrt{3}\,\mathrm{d}y}{\sqrt{(a-y)(y-b)(y-c)}} = \mp \frac{1}{\eta}\,\mathrm{sn}^{-1}\left(\sqrt{\frac{a-q}{a-b}}\,\middle|\, m\right)$$

in which $\eta = \sqrt{(a-c)/12}$ and $m = (a-b)/(a-c)$ and so $a - b = 12\eta^2 m$. Rearranging and using $\mathrm{sn}^2 + \mathrm{cn}^2 = 1$ gives

$$q = b + (a-b)\,\mathrm{cn}^2(\eta(z-z_0)|m)$$

Coefficient of q^2 in $G(q)$ is zero so $a + b + c = 0$. Using this to get an expression for b in terms of η and m gives

$$u(x,t) = V - 4\eta^2(2m-1) + 12\eta^2 m\,\mathrm{cn}^2(\eta(x - x_0 - Vt)|m)$$

The aperiodic (solitary wave/soliton) solution is obtained when $m = 1$:

$$u(x,t) = V - 4\eta^2 + 12\eta^2\,\mathrm{sech}^2(\eta(x - x_0 - Vt))$$

References

- Good RH (2001)

  "Elliptic integrals, the forgotten functions".

  European Journal of Physics, **22**(2), 119–26.

- Meyer KR (2001)

  "Jacobi elliptic functions from a dynamical systems point of view".

  American Mathematical Monthly, **108**, 729–37.

- Byrd PF, Friedman MD (1954)

  Handbook of Elliptic Integrals for Engineers and Physicists, Springer.

  [Contains $>$3000 integrals related to elliptic integrals and functions.]

THANK YOU FOR YOUR ATTENTION