# Maximal commutative subalgebras of Grassmann Algebra

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# References

1. M. Domoskos and M. Zubor. Commutative subalgebras of the Grassmann algebra, *J. Algebra Appl.* **14**(8):1550125, 13, (2015).

2. V. Bovdi and H-H. Leung. Maximal commutative subalgebras of a Grassmann algebra, to appear in *J. Algebra Appl.* (arXiv:1803.03457v1 [math.RA])

Let $V$ be a vector space of dimension $n$ over a field $\mathbb{F}$ (char $\mathbb{F} \neq 2$), with a basis $\{v_1, v_2, ..., v_n\}$.

**Definitions**

The *Grassmann Algebra* (or *Exterior Algebra*) $G(n) := E^{(n)}$ of $V$ is the (associative) $\mathbb{F}$-algebra given by:

$$G(n) = \mathbb{F} < v_1, ..., v_n \mid v_i v_j = -v_j v_i, \ (1 \leq i, j \leq n) >$$

Note:

$$\mathrm{Dim}(G(n)) = \binom{n}{0} + \binom{n}{2} + ... + \binom{n}{n} = 2^n.$$

# Properties of Grassmann Algebra

Example:

Basis elements of $G(4)$:
$1$,
$v_1, v_2, v_3, v_4$,
$v_1 v_2, v_1 v_3, v_1 v_4, v_2 v_3, v_2 v_4, v_3 v_4$,
$v_1 v_2 v_3, v_1 v_2 v_4, v_1 v_3 v_4, v_2 v_3 v_4$,
$v_1 v_2 v_3 v_4$.

Example: $v_1 v_2 v_1 = -v_1 v_1 v_2 = 0$.

$\operatorname{Dim} G(4) = 2^4 = 16$.

Notes:

(1) $v_i^2 = 0$, for all $i = 1, ..., n$.

(2) If $I = \{i_1, ..., i_k\} \subset \{1, ..., n\}$, define $v_I$ by $v_I = v_{i_1} v_{i_2} ... v_{i_k}$. Let $a = v_I$, $b = v_J$, then

$$ab = (-1)^{|I||J|} ba.$$

(Example: $a = v_1 v_2$, $b = v_3 v_4$, then

$$\begin{aligned} ab &= (v_1 v_2)(v_3 v_4) = -v_1 v_3 v_2 v_4 \\ &= v_3 v_1 v_2 v_4 = -v_3 v_1 v_4 v_2 \\ &= v_3 v_4 v_1 v_2 = ba \end{aligned}$$

(3) $v_I$ and $v_J$ anti-commutes iff $|I|$ and $|J|$ are odd. They commute otherwise.

# Our question

**Question**

Can we understand the structure of maximal commutative subalgebras of $G(n)$?

Note:

(1) Let $G_0 = \text{span} < v_I \mid |I| \text{ is even} >$. It is a commutative subalgebra in $G(n)$.

(2) A maximal commutative subalgebra in $G(n)$ must contain $G_0$. (Hence, it has dimension at least $2^{n-1}$.)

(3) A max. coommutative subalgebra in $G(n)$ can be written as $G_0 \oplus A_1'$, where $A_1'$ is spanned by some $v_J$, $|J|$ is odd.

# Commutative subalgebras of $G(n)$

Our goal: To understand the the structure of $A'_1$.

Note:

(1) We only need to consider monomials, $v_I$, where $|I|$ is odd.
(2) Since $v_I v_J = -v_J v_I$ when $|I|$ and $|J|$ are odd,

$$v_I v_J = v_J v_I \iff v_I v_J = 0$$

$$v_I v_J = 0 \iff I \cap J \neq \emptyset$$

Example: $v_I = v_1 v_2 v_3$, $v_J = v_2$,

$$v_I v_J = v_1 v_2 v_3 v_2 = -v_1 v_2 v_2 v_3 = 0$$

$[n] := \{1, ..., n\}$.
$\mathcal{P}(n)$:=set of all subsets of $[n]$.

**Definitions**

Let $\mathcal{S} \subset \mathcal{P}(n)$ be a collection of subsets of odd size in $[n]$. It is *commutative* if $S_1 \cap S_2 \neq \emptyset$ for any $S_1, S_2 \in \mathcal{S}$.

Example: $n = 4$.
(1) $\mathcal{S}_1 = \{\{1,2,3\},\{1,3,4\},\{1\}\}$ is commutative. ($\{1\}$ is a common intersection for all subsets.)

(2) $\mathcal{S}_2 = \{\{1,2,3\},\{1,3,4\}\}$ is commutative.

(3) $\mathcal{S}_3 = \{\{1\},\{2\},\{1,2,3\},\{1,3,4\}\}$ is not commutative.

($A = G_0 \oplus A_1'$ is an algebra:
If $v_I \in G_0$ and $v_J \in A_1'$, then $v_I v_J \in A$.)

## Definitions

Let $\mathcal{S} \subset \mathcal{P}(n)$ be a commutative system of subsets of odd size in $[n]$. It is *algebraic* if $S \cup S_1 \in \mathcal{S}$ for any $S$ of even size, $S_1 \in \mathcal{S}$ and $S \cap S_1 = \emptyset$.

(1) $\mathcal{S}_1 = \{\{1,2,3\},\{1,3,4\},\{1\}\}$ is commutative but not algebraic. ($\{1,2,4\} = \{1\} \cup \{2,4\} \notin \mathcal{S}_1$.)

(2) $\mathcal{S}_2 = \{\{1\},\{1,2,3\},\{1,3,4\},\{1,2,4\}\}$ is commutative and algebraic.

# From algebra to combinatorics

## Definitions

Let $\mathcal{S} \subset \mathcal{P}(n)$ be a commutative algebraic system of subsets of odd size in $[n]$. It is *maximal* if $S$ has odd size and $S \notin \mathcal{S}$, then there exists $S_1 \in \mathcal{S}$ such that $S \cap S_1 = \emptyset$.)

Example: $n = 4$,

(1) $\mathcal{S}_1 = \{\{1,2,3\},\{1,3,4\},\{1,2,4\}\}$ is commutative, algebraic, but not maximal. ($\{1\} \notin \mathcal{S}_1$, and it intersects with all members in $\mathcal{S}_1$.)

(2) $\mathcal{S}_2 = \{\{1\},\{1,2,3\},\{1,3,4\},\{1,2,4\}\}$ is commutative, algebraic and maximal.

(3) $\mathcal{S}_3 = \{\{1,2,3\},\{1,3,4\},\{1,2,4\},\{2,3,4\}\}$ is commutative, algebraic and maximal.

# From algebra to combinatorics

Let $N$ be an subalgebra of $G(n)$. We define

$$\nabla_N := \{I \subseteq [n] \mid v_I \in N,\ |I| = odd\}.$$

Let $T$ be a maximal commutative algebraic system in $[n]$. We define:

$$N_T := \operatorname{span}\langle v_I \mid I \in T\rangle.$$

We have the following lemma:

**Lemma**

If $M$ is a maximal commutative subalgebra of $G(n)$, then $\nabla_M$ is a maximal commutative algebraic system.
If $T$ is a maximal commutative algebraic system, then $G_0 \oplus N_T$ is a maximal commutative subalgebra in $G(n)$.

# Examples of max. comm. algebraic system

$\mathcal{P}_n(i)$:= collection of subsets of size $i$ in $[n]$.

If $n = 4k$,
the following $\mathcal{S}_1$ is a max. commutative algebraic system.

$$\mathcal{S}_1 = \cup_{i \text{ is } odd, 2k+1 \leq i \leq 4k-1} \mathcal{P}_n(i).$$

The following $\mathcal{S}_2$ is a max. commutative algebraic system: for $i \in [n]$,

$$\mathcal{S}_2(i) = \{S \mid |S| = odd, i \in S\}.$$

Note: We have max. comm. subalgebras $G_0 \oplus N_{\mathcal{S}_1}$ and $G_0 \oplus N_{\mathcal{S}_2(i)}$.

$$\dim G_0 \oplus N_{\mathcal{S}_1} = \dim G_0 \oplus N_{\mathcal{S}_2(i)}$$

but they are not isomorphic.

# Examples for $n = 4$

$\dim G(4) = 2^4 = 16.$

$$\mathcal{S}_1 = \mathcal{P}_4(3) = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}.$$
$$\mathcal{S}_2(1) = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1\}\}.$$
$$\mathcal{S}_2(4) = \{\{2,3,4\},\{1,2,4\},\{1,3,4\},\{4\}\}.$$

$\dim(G_0 \oplus N_{\mathcal{S}_1}) = \dim(G_0 \oplus N_{\mathcal{S}_2(1)}) = \dim(G_0 \oplus N_{\mathcal{S}_2(4)}) = 8+4 = 12,$

Note: $G_0 \oplus N_{\mathcal{S}_1}$ is not isomorphic to $G_0 \oplus N_{\mathcal{S}_2(1)}$.
But, $G_0 \oplus N_{\mathcal{S}_2(1)}$ is isomorphic to $G_0 \oplus N_{\mathcal{S}_2(4)}$.

If $n = 4k+2$,
the following $\mathcal{S}_1$ is a max. commutative algebraic system.
For $j \in [n]$,

$$\mathcal{S}_1 = \left(\cup_{i \text{ is } odd, 2k+3 \leq i \leq 4k+1} \mathcal{P}_n(i)\right) \cup \{U \mid U \in \mathcal{P}_n(2k+1),\ j \in U\}.$$

The following $\mathcal{S}_2$ is a max. commutative algebraic system: for $i \in [n]$,

$$\mathcal{S}_2(i) = \{S \mid |S| = odd, i \in S\}.$$

# Examples for $n = 6$

$\dim G(6) = 2^6 = 64.$

$$\mathcal{S}_1(1) = \mathcal{P}_6(5) \cup \{U \mid 1 \in U, |U| = 3\}.$$

$\mathcal{S}_2(3)$ contains all subsets of odd size which contains 3.

$$\dim(G_0 \oplus N_{\mathcal{S}_1(1)}) = \dim(G_0 \oplus N_{\mathcal{S}_2(3)}) = 32 + 16 = 48,$$

but they are not isomorphic.

# For the case: $n$ is even

**Theorem (Domokos and Zubor (2015))**

If $n$ is even, then every maximal commutative subalgebra $M(n)$ of $G(n)$ has dimension $2^{n-1} + 2^{n-2} = 3(2)^{n-2}$.

Note: Not all max. commutative subalgebras of $G(n)$ ($n$ is even) are isomorphic.

# For the case: $n$ is odd

Remaining question:

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| *Find all max. comm. subalgebras of $G(n)$ when $n$ is odd.* |

For any odd $n$, the following $\mathcal{S}$ is a max. commutative algebraic system: for $i \in [n]$,

$$\mathcal{S}(i) = \{S \mid |S| = odd, i \in S\}.$$

Fact:

$$|\mathcal{S}(i)| = 2^{n-2}.$$

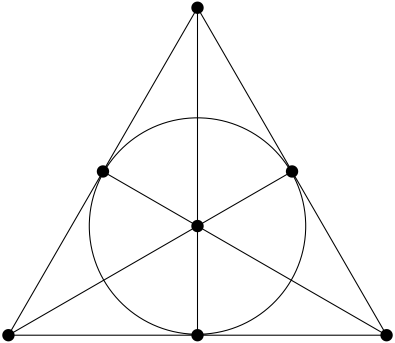
$$\dim(G_0 \oplus N_{\mathcal{S}(i)}) = 2^{n-1} + 2^{n-2} = 3 \cdot 2^{n-2}$$

and $G_0 \oplus N_{\mathcal{S}(i)}$ are all isomorphic for different $i$.

# New constructions based on Fano Plane

Goal: Construct new maximal commutative algebraic systems for $[n]$ when $n$ is odd.

Idea: Fano plane (by Gino Fano (1871-1952))



# Fano System

## Definition

*The Fano Set* contains the following subsets of the set [7]:

$$A_1 = \{1,2,5\}, A_2 = \{1,3,6\}, A_3 = \{1,4,7\}, A_4 = \{2,3,7\},$$

$$A_5 = \{3,4,5\}, A_6 = \{5,6,7\}, A_7 = \{2,4,6\}.$$

Note:
(1) It is a commutative and maximal in [7] among subsets of size 3.
(2) For $n = 7$, $m = 3$, by Erdos-Ko-Rado Theorem, the maximum dimension of a maximal commutative system in [7] of size 3 is 15. And the minimum dimension of a maximal commutative system is 7 provided by Fano Set.

Recall:

**EKR's theorem**

Let $k, n$ be two integers such that $2k \leq n$. Suppose $\mathcal{S}$ is a family of subsets of size $k$ in $[n]$ such that each pair of subsets has non-empty intersection, then the number of subsets in $A$ is $\leq \binom{n-1}{k-1}$.

The maximal family of subsets can be constructed by picking all subsets of size $k$ which contain a common point.

Example: $n = 5$, $k = 2$.
$\mathcal{S} = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\}\}$. $|\mathcal{S}| = \binom{5-1}{2-1} = \binom{4}{1} = 4$.

# An open question in finite combinatorics

How can we construct the *minimum* family of subsets of size $k$ Iwhich are *maximal commutative* such that each pair of subsets has non-empty intersection?

Example: $n = 7, k = 3$,
Fano set provides such an example.

# Generalized Fano System for $n = 4k + 7$

Let $n \geq 2$.

Our idea: Construct a *maximal commutative system* in $[4k + 7]$ of size $2k + 3$ with dimension (significantly) less than the dimension $\binom{4k+6}{2k+2}$ provided by EKR's theorem.
We call it a *generalized Fano system*.

| |
|---|
| Question (Q1): Does the generalized Fano system have minimum dimension among all maximal commutative systems in $[4k + 7]$ for subsets of size $2k + 3$? |

# Rephrasing Q1

Let $k \geq 2$. Let $n = 4k + 7$.

Let $\mathcal{S}$ be an intersecting family of subsets of size $2k + 3$ in $[n]$, define

$$Spec(k, 2) = \min\{|\mathcal{S}| \mid \mathcal{S} \text{ is maximal commutative.}\}$$

By EKR's theorem:

$$Spec(k, 2) \leq \binom{4k+6}{2k+2}.$$

# Rephrasing Q1

Let $\mathcal{A}_k$ be the generalized Fano's system (with subsets of size $2k+3$ in $[n]=[4k+7]$).

As $k \to \infty$,

$$\frac{|\mathcal{A}_k|}{\binom{4k+6}{2k+2}} \to 0.5625$$

Hence,

$$\lim_{k\to\infty} \frac{Spec(k,2)}{\binom{4k+6}{2k+2}} \leq 0.5625.$$

# Rephrasing Q1

**Conjecture 1 (Q1)**

Does the generalized Fano system $\mathcal{A}_k$ have the minimum dimension among all (maximal commutative) intersecting family of subsets of size $2k+3$ in $[4k+7]$?

**Conjecture 2**

Can we find $\lambda$ such that $0 < \lambda < 0.5$ such and

$$\lim_{k \to \infty} \frac{Spec(k,2)}{\binom{4k+6}{2k+2}} < \lambda?$$

# A conjecture for the case $n = 4k + 3$

**Conjecture (Domoskos and Zubor (2015))**

If $n = 4k + 3$, then every maximal commutative subalgebras has dimension $2^{n-1} + \Sigma_{l=k}^{2k} \binom{n}{2l+3} + |\mathcal{S}|$ where $\mathcal{S}$ is a maximal commutative system of minimum dimension.

**Theorem (Bovdi and Leung (2018))**

If $n = 4k + 3$, then the conjecture is false provided that Q1 is correct.

# A conjecture for the case $n = 4k + 1$

**Conjecture (Domoskos and Zubor (2015))**

If $n = 4k + 1$, all maximal commutative subalgebras of $G(n)$ has dimension at least $3 \cdot 2^{n-2}$.

**Theorem (Bovdi and Leung (2018))**

If $n = 4k + 1$ and $17 \leq n \leq 997$, there exists maximal commutative subalgebras of $G(n)$ with dimension less than $3 \cdot 2^{n-2}$.

**Conjecture 3**

The theorem can be extended to all $n$ such that $n \geq 17$.

Thank you!