Mahidol University International College

Algorithms of Linear Programming Part 2

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November 22, 2017

Outline of Talk

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The Ellipsoid Method and Algorithm

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Review: The Linear Programming Problem

Definition

A **linear programming problem** (LP) is a collection of instances that seeks to identify optimal values (max/min) of functions subject to constraints. An instance of such a problem will consist of a feasible set F and an objective linear function $O : F \to \mathbb{R}$ with the objective being to optimize $O(x)$ over all $x \in F$.

Example:

$$\text{maximize } c^T x$$
$$Ax \leq b$$
$$x \geq 0$$
$$\text{for } x \in \mathbb{R}^n$$

We assume componentwise inequalities.

Review: LP Example

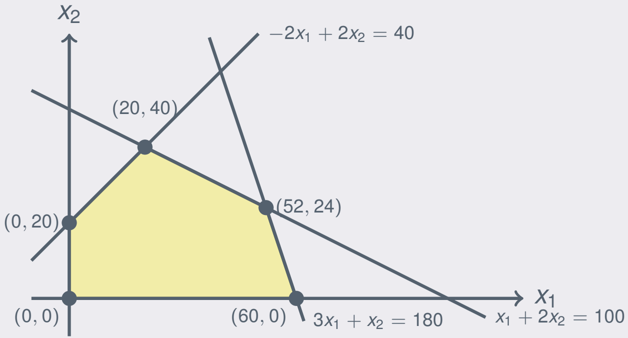
Maximization LP

Maximize $4x_1 + 12x_2$ subject to the constraints:

$$
\begin{aligned}
3x_1 + x_2 &\leq 180 \\
x_1 + 2x_2 &\leq 100 \\
-2x_1 + 2x_2 &\leq 40 \\
x_1 \geq 0, x_2 &\geq 0
\end{aligned}
$$

Review: LP Example

The Feasible Region, F



Review: Simplex Method

The simplex method (Dantzig) maximizes the objective function by computing the objective function at selected corner points of the feasible region until the optimal solution is reached. It begins at the **origin** and moves at each stage to a corner point determined by the variable that will yield the largest increase in *O*. The steps in the algorithm methodically “finds” the point, and avoids problem of cycling through the points (Bland).

Geometrical Background

We start with a ridiculous hypothetical situation.

Scenario: You want to catch a lion in the Sahara. How do you do it?

1. Fence in the Sahara.
2. Split the fenced-in region into two.
3. The lion is in one of the regions, right?
4. Fence in that region.
5. Repeats steps 2-4 until the lion is precisely located or declare it nonexistent (e.g. when the fencing is so small a lion couldn't fit into half of it).

If the Sahara were $\mathbb{R}$ this might sound a little bit like the bisection method. In which case, we would like to describe the ellipsoid method as a more advanced version of the bisection method.

Geometrical Background

We now define an ellipsoid.

Definition

A **general ellipsoid** centered at $z \in \mathbb{R}^n$ can be represented by the set of vectors

$$E = E(z, D) = \{x \in \mathbb{R}^n | (x - z)^T D^{-1} (x - z) \leq 1\}$$

where D is an $n \times n$ positive definite symmetric matrix.

Recall that an $n \times n$ symmetric matrix D is positive definite iff:

- $x' Dx > 0$ for all nonzero vectors $x \in \mathbb{R}^n$.
- D has only real and positive eigenvalues.
- $D = B^T B$ for some nonsingular matrix B.

Note: D is nonsingular and its inverse is also positive definite.

Geometrical Background

1. A basic example in $\mathbb{R}^2$: $5x^2 + 8xy + 5y^2 \leq 1$
2. Every ellipsoid is the image of the unit ball under a bijective affine transformation.
3. $\text{Vol}(E(D,z)) = \sqrt{\det D} \cdot \text{Vol}(S(0,1))$ in $\mathbb{R}^n$
   (by change of variables)

Geometrical Background

Ellipsoids are convex sets.

Definition (Convex Set)

A set A is convex if $(1-t)a + tb \in A$ for all $t \in [0,1]$ and $a, b \in A$.

Examples:

- $\mathbb{R}^n$
- Hypercube of length l in $\mathbb{R}^n$
- Ball of radius r around the origin

Geometrical Background

Generalizing the notion of an ellipsoid in this manner allows us to make the following observation.

Lemma (Farkas' Lemma)

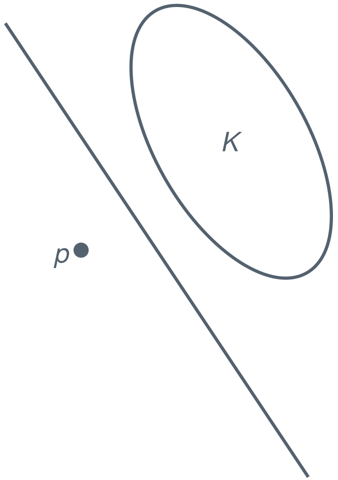
If $K \subset \mathbb{R}^n$ is a convex set and $p \in \mathbb{R}^n$ is a point, then one of the following holds:

1. $p \in K$,
2. *there is a hyperplane* that separates p from K.*

Theorem (Separating Hyperplanes*)

Suppose C and D are nonempty disjoint convex sets, i.e., $C \cap D = \emptyset$. Then there exist $a \neq 0$ and b such that $a'x \leq b$ for all $x \in C$ and $a'x \geq b$ for all $x \in D$. In other words, the affine function $a'x - b$ is nonpositive on C and nonnegative on D. The hyperplane $\{x | a'x = b\}$ is called a separating hyperplane for the sets C and D, or is said to separate the sets C and D.

Geometrical Background



Recall our hypothetical situation. That, and the previous discussion motivates the following concept.

Separation Oracle

The separation oracle is a decision making process for a convex set K that allows us to decide whether a point $p \in K$. If not, then it will return a hyperplane separating p from K.

oracle (n.): A person or agency considered to give wise counsel or prophetic predictions or precognition of the future, inpspired by the gods.

Lӧwner-John Ellipsoids

Theorem

For every convex set $K \subset \mathbb{R}^n$ there exists a unique ellipsoid E of minimal volume containing K. Moreover, K contains the ellipsoid obtained from E by shrinking it from its center by a factor of n. Let us call the minimum volume ellipsoid containing a convex set K the ***Lӧwner-John ellipsoid of*** *K.*

Ellipsoid Method

Let $E = E(z, D)$ be an ellipsoid in $\mathbb{R}^n$ and let a be a nonzero n-vector. Define the pair $(\bar{z}, \bar{D})$ to be

$$\bar{z} = z - \frac{1}{n+1}\frac{Da}{\sqrt{a'Da}}$$

$$\bar{D} = \frac{n^2}{n^2-1}\left(D - \frac{2}{n+1}\frac{(Da)(Da)'}{a'Da}\right).$$

Theorem

The matrix $\bar{D}$ is symmetric and positive definite and thus $E' = E(\bar{z}, \bar{D})$ is an ellipsoid. Let H^- be the half space $\{x \in \mathbb{R}^n | a'x \leq a'z\}$. Then,

1. $(E \cap H^-) \subset E'$
2. $Vol(E') < e^{-1/(2(n+1))} Vol(E)$

Observe that the second result shows that the ellipsoids decrease in volume with each update of $(\bar{z}, \bar{D})$.

Ellipsoid Method

Let's try one iteration. Suppose

$$z = (0, 0) \text{ and } D = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}.$$

Find $\bar{z}$ and $\bar{D}$ and then graph both $E(z, D)$ and $E(\bar{z}, \bar{D})$.

Ellipsoid Method

- INPUTS:
  - A convex set P.
  - A number v for which $\mathrm{Vol}(P) > v$.
  - An initial ball $E_0 = E(x_0, r^2 I)$ with finite volume and $P \subset E_0$.
- OUTPUT:
  - A feasible point $x^* \in P$, else a statement saying that P is empty.
- Pseudocode:
  - While the volume of $E_i > v$, we denote the center of E_i as p.
  - Apply the Separation Oracle on p and P.
  - If the answer is 'yes' take a separating hyperplane H and let E_{i+1} be the minimum volume ellipsoid containing $E_i \cap H^+$. This was described in the previous theorem. Then let $i = i + 1$ and allow it to loop.
  - Otherwise return 'no'.

LP Problems using the Ellipsoid Method

Recall:

Standard LP Problem and its Dual

$\min b^T y$	$\max c^T x$
$A^T y \geq c$	$Ax \leq b$

Theorem (Dantzig)

A standard minimum problem has a solution if and only if its dual problem has a solution. If a solution exists, the standard minimum problem and its dual problem have the same optimal value.

LP Problems using the Ellipsoid Method

How to formulate LP to apply the method

The constraints of the LP problem form the following convex set

$$P = \{x \in \mathbb{R}^n | Ax \leq b\},$$

with corresponding half spaces:

$$H^+ = \{x \in \mathbb{R}^n | a'x \geq a'z\},$$

$$H^- = \{x \in \mathbb{R}^n | a'x \leq a'z\}$$

for a being a column of the matrix A. Strong duality requires that optimal solutions can only occur if and only if the system $b^T y = c^T x$ along with all given contraints is feasible. So if we denote Q to be such a feasible set, then we can apply the ellipsoid method to decide whether Q is nonempty.

LP Problems using the Ellipsoid Method

Some examples to compare both methods with a computer:

Maximize $x_1 + 2x_2$ given

$$\begin{cases} -x_1 - x_2 \leq -2 \\ 3x_1 \leq 4 \\ -2x_1 + 2x_2 \leq 3 \end{cases}$$

A Few Iterations of Ellipsoid

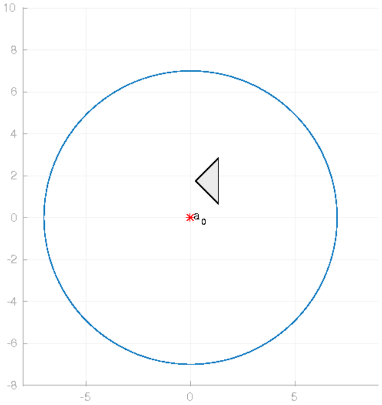


Figure: Initial Ball

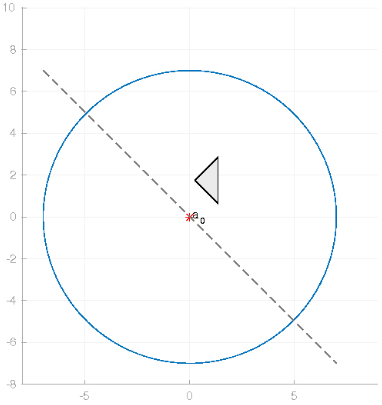


Figure: First Hyperplane

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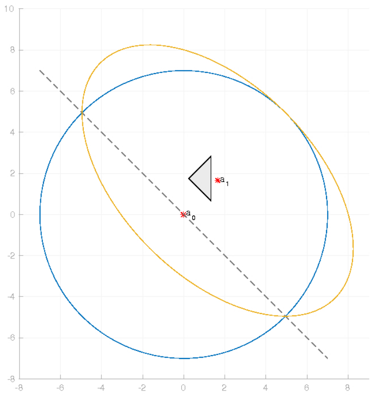

Figure: First Iteration

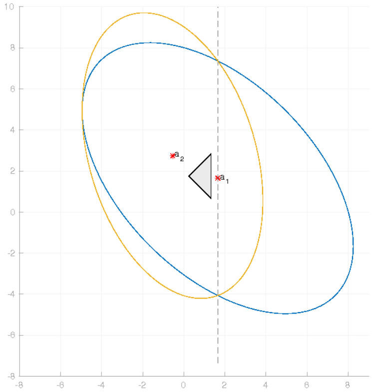

Figure: Second Iteration

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Complexity of Simplex Method

Borgwardt, Haimovich, Smale showed that the simplex runs poylnomial time on average, but we measure the complexity of this algorithm based on a worst case pivoting rule. We first recall the definition of a unit cube in $\mathbb{R}^n$, which has 2^n vertices:

$$0 \leq x_i \leq 1, i = 1, , , n$$

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Worst Case Simplex (Klee-Minty Cube)

Maximize $P = x_n$ subject to the constraints:

$$\left\{ \begin{array}{c} \epsilon \leq x_1 \leq 1 \\ \epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \, i = 2, ..., n \end{array} \right\}$$

for some $\epsilon \in (0, 1/2)$. The feasible set will have 2^n vertices and they can be ordered so that each one is adjacent to and has higher value than the previous one. Klee-Minty showed that **there exists** a pivoting rule such that the tableau changes at least $2^n - 1$ times before termination.

Complexity of Ellipsoid Method

Lemma

The minimum volume ellipsoid surrounding a half ellipsoid can be calculated in polynomial time.

Complexity of Ellipsoid Method

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Theorem (Kachiyan, 1979)

The linear programming feasibility problem with integer data can be solved in polynomial time.

Background and History

People	Technique	Year
Dantzig	Simplex	1947
Klee-Minty	Worst Case Simplex	1972
Shor Yudin Nemirovsky	Ellipsoid Algorithm	1970’s
Khachiyan	Ellipsoid Proof	1979
Karmarkar	Interior Point	1984
Many People	Improvements	1980s to present

Implications

- This algorithm can be applied to **any** problem which has its own corresponding separation oracle (resulting a nice polynomial time algorithm for the corresponding problem).
- It is a difficult problem because of the insistence of a universal polynomial bound for **all** instances of the problem. Thus, it is mainly used for classification and not as useful in practice.
- The more practical interior point method was developed after knowledge that it was possible! Unfortunately requires knowledge of entire constraint system (similar to simplex).
- The fanfare behind Khachiyan's proof was unprecedented in the nonscientific world at that time. Good PR though.

References for Further Reading

1. D. Bertsimas, J. N. Tsitsiklis, *Introduction to Linear Optimization*. Athena Scientific, New Hampshire, USA, 1997.
2. M. Grötschel, L. Lovász, A. Schrijver *Geometric Algorithms and Combinatorial Optimization*. Second Corrected Edition, Springer-Verlag, Germany, 1993
3. C. Papadimitriou, K. Steiglitz, *Combinatorial Optimization, Algorithms and Complexity*. Dover Publications, New York, USA, 1998.

Thank you!