Bijective Proofs with Spotted Tilings

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outline

- examples of bijective proofs
  - binomial coefficients
  - compositions and Fibonacci numbers
- n-color compositions and spotted tilings
  - restricting part size
  - restricting colors

binomial coefficients

What is $\binom{n}{k}$?

Binomial coefficient definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

NO! NO! NO! NO! NO!

Binomial coefficient definition

$\binom{n}{k}$ is the number of ways to choose k out of n objects.

The definition has combinatorial meaning; the formula follows.

We will usually think of choosing k positions to put spots on a length n board.

Binomial coefficient sum

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

How many ways can you put any number of spots on an n-board?

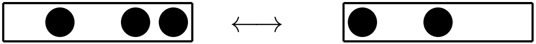
Left hand side: Add up the number of ways to place k spots for all possible values of k.

Right hand side: Go through the n positions and make the binary choice “spot or not” for each one. □

Complementarity

$$\binom{n}{k} = \binom{n}{n-k}$$

Swap "spot or not" for each position:



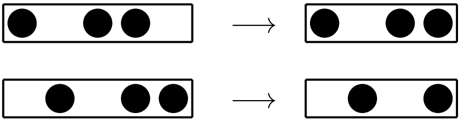
This establishes a one-to-one correspondence between k spot boards and $n - k$ spot boards. □

A combinatorial proof (often done via a bijection) attempts to "show" that a result is true by often visual means rather than symbolic manipulation.

Pascal's triangle rule

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Starting from the left hand side, consider all possible ways to choose k spots in n positions. Remove the nth position (which may or may not have a spot).



If the removed position has no spot, we are left with k spots in $n-1$ positions; there are $\binom{n-1}{k}$ of these.

If the removed position has a spot, we are left with $k-1$ spots in $n-1$ positions; there are $\binom{n-1}{k-1}$ of these.

binomial coefficients

Looking for patterns: We know $\sum_{k=0}^{n} \binom{n}{k}$. What else could we try?

n	$\displaystyle\sum_{k=1}^{n} k\binom{n}{k}$
1	$\binom{1}{1} = 1$
2	$\binom{2}{1} + 2\binom{2}{2} = 4$
3	$\binom{3}{1} + 2\binom{3}{2} + 3\binom{3}{3} = 12$
4	$\binom{4}{1} + 2\binom{4}{2} + 3\binom{4}{3} + 4\binom{4}{4} = 32$
5	$\binom{5}{1} + 2\binom{5}{2} + 3\binom{5}{3} + 4\binom{5}{4} + 5\binom{5}{5} = 80$

A weighted sum

$$\sum_{k=1}^{n} k\binom{n}{k} = n\,2^{n-1}$$

$$\begin{aligned}\sum_{k=1}^{n} k\binom{n}{k} &= \sum_{k=1}^{n} \frac{k\,n!}{k!(n-k)!} = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} \\ &= n\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} = n\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} \\ &= n\sum_{k=1}^{n} \binom{n-1}{k-1} = n\,2^{n-1} \qquad \square\end{aligned}$$

A weighted sum

$$\sum_{k=1}^{n} k \binom{n}{k} = n\,2^{n-1}$$

How many ways can you choose a committee where one member is designated as chair?

Left hand side: For each size k, choose of committee, then choose one of the k members to be chair.

Right hand side: Choose one of the n people to be chair, form a committee from the remaining $n-1$ people. □

binomial coefficients

Looking for patterns: We know $\sum_{k=0}^{n}\binom{n}{k}$ and $\sum_{k=0}^{n}k\binom{n}{k}\ldots$

n	$\displaystyle\sum_{k=0}^{n}\binom{n}{k}^2$
1	$\binom{1}{0}^2+\binom{1}{1}^2=2$
2	$\binom{2}{0}^2+\binom{2}{1}^2+\binom{2}{2}^2=6$
3	$\binom{3}{0}^2+\binom{3}{1}^2+\binom{3}{2}^2+\binom{3}{3}^2=20$
4	$\binom{4}{0}^2+\binom{4}{1}^2+\binom{4}{2}^2+\binom{4}{3}^2+\binom{4}{4}^2=70$
5	$\binom{5}{0}^2+\binom{5}{1}^2+\binom{5}{2}^2+\binom{5}{3}^2+\binom{5}{4}^2+\binom{5}{5}^2=252$

binomial coefficients

Sum of squared terms

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Using complementarity, rewrite the left hand side:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$

Right hand side: Number of ways to put n spots in $2n$ positions.

Left hand side: For each feasible k, number of ways to put k spots in the first n positions and $n-k$ spots in the second n positions. □

compositions

A composition of n is an ordered tuple of positive integers with sum n.

Compositions of 3 are $\{(3), (2,1), (1,2), (1,1,1)\}$, write $C(3) = 4$.

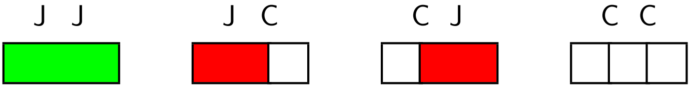
n-board visualization from MacMahon's *Combinatory Analysis*.

For clarity, we use the coloring scheme of Cuisenaire rods.

Number of compositions

$$C(n) = 2^{n-1}$$

The n-board has $n-1$ possible places to either join a square to the part to the left or make a cut and start a new part.



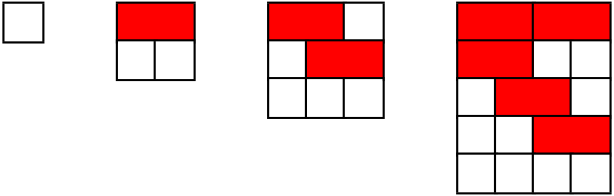
Number of compositions with k parts

$$C(n,k) = \binom{n-1}{k-1}$$

A k part composition has $k-1$ cuts.

Several sequences with $a(n) \leq 2^{n-1}$ count compositions with various restrictions.

Consider compositions where the only allowed parts are 1 and 2.

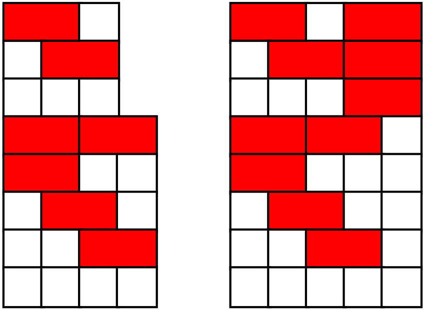


1, 2, 3, 5, . . . looks like Fibonacci. (Actually known by Virahāṅka circa 600 CE in association with Sanskrit poetry; Singh 1985 *Historia Mathematica*.)

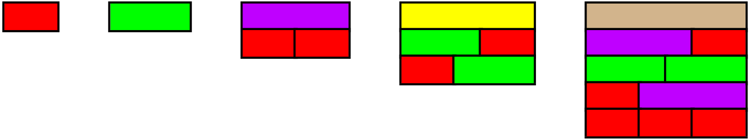
$F(0) = 0, F(1) = 1, F(2) = 1, F(3) = 2, F(4) = 3, F(5) = 5;$
recurrence $F(n) = F(n-1) + F(n-2)$.

Gopāla–Hemachandra, Fibonacci

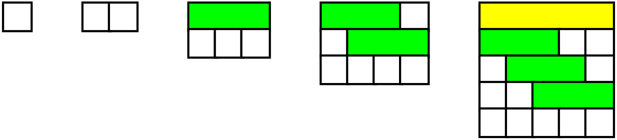
$$C[1,2](n) = F(n+1)$$



Consider compositions where 1 is not allowed as a part.

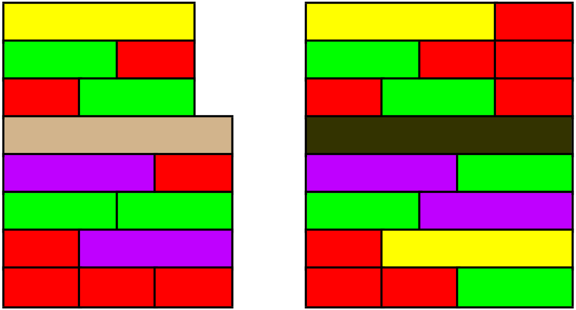


Where only odd numbers are allowed as parts.



Cayley 1876

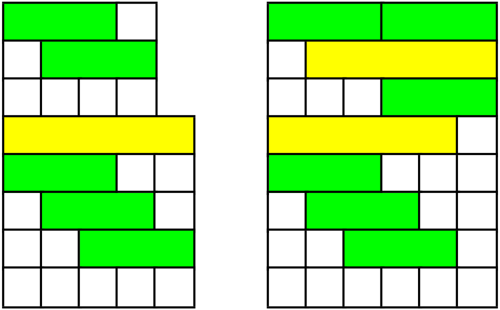
$$C[\text{no } 1](n) = F(n-1)$$



compositions

Hoggatt–Lind 1965 ??

$$C[\text{odds}](n) = F(n)$$

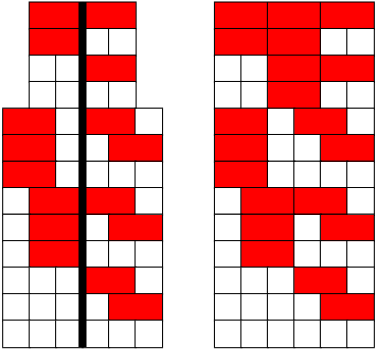


compositions

$1^1 + 2^2 = 5,\ 2^2 + 3^2 = 13,\ 3^2 + 5^2 = 34,\ 5^2 + 8^2 = 89,\ \ldots$

A Fibonacci square identity

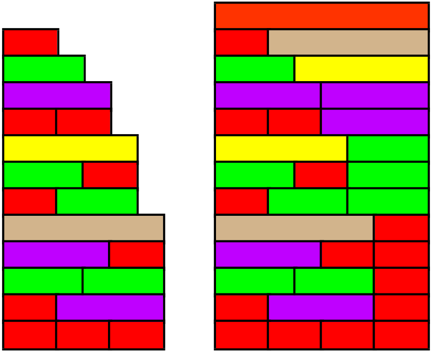
$$(F(n))^2 + (F(n+1))^2 = F(2n+1)$$



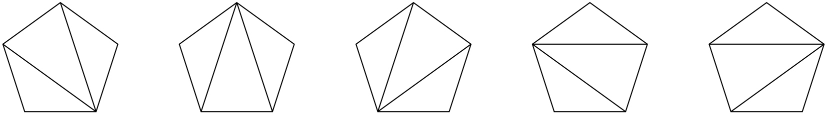
$1+1+2=4,\ 1+1+2+3=7,\ 1+1+2+3+5=12,\ \ldots$

A Fibonacci sum identity

$$\sum_{k=1}^{n} F(k) = F(n+2) - 1$$



Catalan numbers: 2015 Richard Stanley book discusses 214 objects counted by 1, 2, 5, 14, 42, . . .



$$\begin{vmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{vmatrix} \quad \begin{vmatrix}1 & 3\\ 2 & 4\\ 5 & 6\end{vmatrix} \quad \begin{vmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{vmatrix} \quad \begin{vmatrix}1 & 2\\ 3 & 5\\ 4 & 6\end{vmatrix} \quad \begin{vmatrix}1 & 3\\ 2 & 5\\ 4 & 6\end{vmatrix}$$

Combinatorial proofs also used for permutations, graph theory, lattice paths, integer partitions, etc. etc. etc.

An *n*-color composition of *n* is a composition of *n* where there are *k* different copies of each part *k* available as parts.

The *n*-color compositions of 3 are

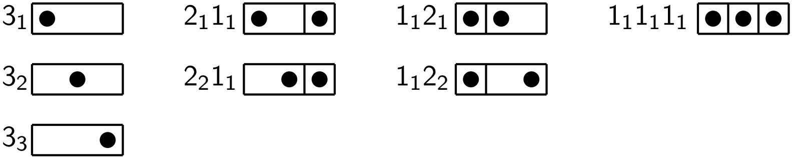
$$\{(3_1), (3_2), (3_3), (2_1, 1_1), (2_2, 1_1), (1_1, 2_1), (1_1, 2_2), (1_1, 1_1, 1_1)\}.$$

Introduced by Agarwal in 2000, following 1987 Agarwal & Andrews partitions with "*n* copies of *n*."

Subject of about a dozen papers: Agarwal, Geetika Narang, Yu-hong Guo. Mirroring questions on standard compositions, primarily on counting *n*-color compositions with restrictions on part sizes.

n-color compositions

"Spotted tiling" visualization (H. *Integers* 2013): part k_i corresponds to a $1 \times k$ rectangle with a spot in position i.

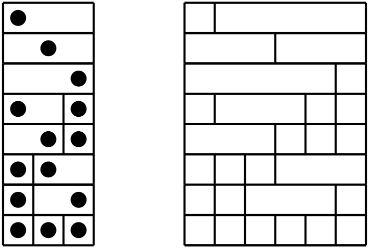


Nice tool for bijections and combinatorial proofs of recurrence relations & direct formulas.

n-color composition count

Agarwal 2000

The number of n-color compositions of n, $CC(n) = F(2n)$.



Think of spots being at half-integer positions. Bars and spots alternate, distances $k + \frac{1}{2}$. Double the distances. Reversible: In odd-part compositions, bars alternate even and odd positions.

H. 2013

The number of n-color compositions of n with only odd parts is given by the recurrence

$$a(n) = a(n-1) + 2a(n-2) + a(n-3) - a(n-4)$$

with initial values $0, 1, 1, 4$ (OEIS A119749).

Prove by a bijection showing

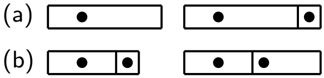
$$a(n) + a(n-4) = a(n-1) + 2a(n-2) + a(n-3).$$

bijection for odd part n-color compositions

From $a(n-1) + 2a(n-2) + a(n-3)$:

(a) Given an n-color composition counted by $a(n-1)$, add 1_1.

(b) For the first set of n-color composition counted by $a(n-2)$, replace the final part k_j with $(k+2)_j$.



Aiming for $a(n) + a(n-4)$. So far odd part n-color compositions ending in 1_1 and k_j with $j \leq k-2$.

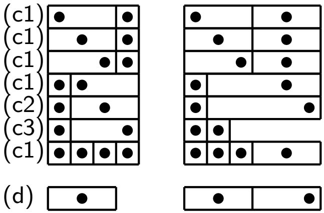
bijection for odd part n-color compositions

Continuing from $a(n-1) + 2a(n-2) + a(n-3)$:

(c) For the second set counted by $a(n-2)$,

- (c1) if the last part is k_1, replace it with $(k+2)_{k+1}$,
- (c2) if the last part is k_2, replace it with $(k+2)_{k+2}$,
- (c3) if the last part is k_j with $j \geq 3$, replace it with $(k-2)_{j-2}$.

(d) Given an n-color composition counted by $a(n-3)$, add 3_3.



Gives remaining counted by $a(n)$ and $a(n-4)$ (c3).

restricted color n-color compositions

Rather than restricting part size, restrict colors.

Motivation:

- Posit n-color compositions as a useful class of combinatorial objects.
- Provide new combinatorial interpretations for integer sequences. (Restricted compositions require $a(n) \leq 2^{n-1}$, restricted n-color compositions $a(n) \leq F(2n) \sim 2.618^{n-1}$.)
- Provide connections to various other combinatorial objects.
- Provide tools for finding recurrence relations, generating functions, direct formulas.

Recurrences for restricted color n-color compositions

For all, $a(0) = 1$ (the empty composition) and $a(k) = 0$ for $k < 0$.

- The number of compositions with allowed colors $\{c_i\}$ is

$$a(n) = a(n-1) + \sum_i a(n - c_i)$$

- The number of compositions with forbidden colors $\{d_i\}$ is

$$a(n) = 3a(n-1) - a(n-2) + \sum_i (-a(n - d_i) + a(n - d_i - 1))$$

- The number of compositions with colors congruent to $\{m_i\}$ modulo m chosen such that $1 \leq m_i \leq m$ for each i is

$$a(n) = a(n-1) + (\sum_i a(n - m_i)) + a(n - m) - a(n - m - 1)$$

The many special cases, we have bijections to other objects and/or direct formulas with combinatorial proofs.

- Allowed colors
  - Any one color
  - Color pairs k and $k+1$
  - Color pairs 1 & 3, 1 & 4
  - Colors $1, \ldots, k$
- Prohibited colors
  - Any one color not allowed
  - No colors $1, \ldots, k$
- Modular colors
  - odd colors
  - even colors

only colors 1 & 2

A001333: $1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, \ldots$

recurrence: $a(n) = 2a(n-1) + a(n-2)$

Bijection to length $n-1$ ternary words avoiding 02 and 20:

Consider the $n-1$ joins or cuts in a length n spotted tiling.

- J in color 1 part $\longleftrightarrow$ 0
- J in color 2 part $\longleftrightarrow$ 2
- C $\longleftrightarrow$ 1

J C J J C J J J C C J C J J J

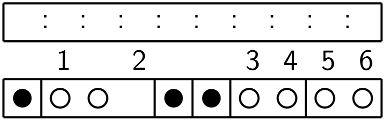
2 1 0 0 1 2 2 2 1 1 2 1 0 0 0

A001333: $1, 3, 7, 17, 41, 99, 239, 577, 1393, \ldots$

recurrence: $a(n) = 2a(n-1) + a(n-2)$

Direct formula: $a(n) = \sum_{m=0}^{\lfloor n/2 \rfloor} 2^m \binom{n}{2m}$

Consider an $n = 10$ and $m = 3$ term.



Longer parts determined by $(2i-1, 2i)$ with 2 color choices each, remaining are 1_1.

no color 2

A034943: $1, 2, 5, 12, 28, 65, 151, 351, 816, \ldots$

recurrence: $a(n) = 3a(n-1) - 2a(n-2) + a(n-3)$

Bijection to length $3n+2$ (normal) compositions with parts congruent to 2 modulo 3:

Write $\cdot 2$ for a length 2 part that can join to the left to become a longer part (but not on the right), similarly $3\cdot$.

- $1_1 \longleftrightarrow 3\cdot$
- $k_1 \longleftrightarrow \cdot 2, 2, 3k-4$ for $k \geq 2$
- $k_j \longleftrightarrow \cdot 2, 3j-4, 3k-3j+2$ for $k \geq 2$ and $j \geq 2$.
- End with a $\cdot 2$ and join parts where both sides are open.

$$(2_1, 1_1, 3_3, 5_4, 1_1)$$

$$\cdot 2, 2, 2 \quad 3\cdot \quad \cdot 2, 5, 2 \quad \cdot 2, 8, 5 \quad 3\cdot \quad \cdot 2$$

$$(2, 2, 2, 5, 5, 2, 2, 8, 5, 5)$$

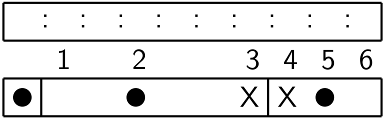
no color 2

A034943: 1, 2, 5, 12, 28, 65, 151, 351, 816, ...

recurrence: $a(n) = 3a(n-1) - 2a(n-2) + a(n-3)$

Direct formula: $a(n) = \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n+m}{3m}$

Consider an $n = 8$ and $m = 2$ term.



so $(1_1, 5_3, 2_1)$. Longer parts determined by $(3i-2, 3i-1, 3i)$: spot at $3i-1$, first position removed if $3i-2$ & $3i-1$ adjacent, else last position removed.

no color 3

A191652: $1, 3, 7, 18, 45, 113, 283, 709, 1775, \ldots$

recurrence: $a(n) = 3a(n-1) - a(n-2) - a(n-3) + a(n-4)$

Recurrence not listed on OEIS!

Interpretation: "Number of n-step two-sided prudent self-avoiding walks ending on the top side of their box." (2001 *Phys. Rev. E*, then Bousquet-Mélou 2010 *JCTA*.)