Surprising Connections Between Distinct and Complete Partitions

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Overview

Outline:

- Integer partitions: distinct (Euler); perfect (MacMahon), complete (Park), k-tight (new)
- Number theory concepts for partitions (Schneider)
- Connections via matrices, bijections, generating functions

Joint work in progress with George Andrews (Pennsylvania State University) and George Beck (Wolfram).

A partition of n is an unordered collection of positive integers whose sum is n.

$$P(4) = \{(1^4), (1,1,2), (2,2), (1,3), (4)\}; \qquad p(4) = 5$$

Write $\lambda \in P(n)$ as $\lambda = (\lambda_1, \ldots, \lambda_t)$. Related notation: $\lambda \vdash n$ and $|\lambda| = n$. The λ_i are the parts, given here in non-decreasing order. The length of λ is the number of parts t.

Partitions

Euler did initial substantial work on partitions. Generating function:

$$(1+x^1+x^{1+1}+\cdots)(1+x^2+x^{2+2}+\cdots)(1+x^3+x^{3+3}+\cdots)\cdots$$
$$=1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+\cdots$$

$$5x^4 = x^{1+1+1+1} + x^{1+1}x^2 + x^{2+2} + x^1x^3 + x^4$$

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

For partitions with distinct parts,

$$\prod_{k=1}^{\infty}(1+x^k) = 1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+\cdots$$

Perfect partitions

MacMahon in *Combinatory Analysis* (1916) defined perfect partitions: $\lambda \vdash n$ with, for each $1 \leq k \leq n$, a unique subpartition whose sum is k.

The perfect partitions of 7 are

$$(1^7), (1, 2, 2, 2), (1, 1, 1, 4), (1, 2, 4).$$

Used in his analysis of Latin squares.

Number of perfect partitions of n sequence is OEIS A002033.

From a journal problem ...

In the summer 1960 issue of *The American Mathematical Monthly*:

E 1424. *Proposed by V. E. Hoggatt and Charles King, San Jose State College*

A set of positive integers W_i, $i = 1, 2, \ldots$, is *complete* if for each positive integer N there exists a subset W_{i_j}, $j = 1, \ldots, k$, of set W_i such that $N = \sum_{j=1}^{k} W_{i_j}$.

1. Show that the set of Fibonacci numbers F_i ($F_{i+2} = F_{i+1} + F_i$, $F_1 = F_2 = 1$) is complete.
2. Show that if any one Fibonacci number is deleted from the set F_i, the set is still complete, but the deletion of two Fibonacci numbers renders the set incomplete.

One solver showed many related results, leading to

John Lawrence Brown, Jr. (1961), Note on complete sequences of integers, *American Mathematical Monthly* 68: 557–560.

which mostly considers infinite sequences of integers (e.g., the Fibonacci numbers).

Brown, Theorem 1

Let $\{f_i\}_{i=1}^{\infty}$ be a nondecreasing sequence of positive integers with $f_1 = 1$. Then $\{f_i\}$ is complete if and only if $f_{p+1} \leqq 1 + \sum_{i=1}^{p} f_i$ for $p = 1, 2, \ldots$.

Combining completeness and partitions

A perfect partition $\lambda \vdash n$ requires, for each $1 \leq k \leq n$, a unique subpartition whose sum is k. Drop uniqueness to get...

A complete partition $\lambda \vdash n$ requires, for each $1 \leq k \leq n$, at least one subpartition whose sum is k.

SeungKyung Park (1998), Complete partitions, *Fibonacci Quarterly* 36: 354–360.

The complete partitions of 7 are

$$(1^7), (1,2,2,2), (1,1,1,4), (1,2,4);$$
$$(1^5,2), (1,1,1,2,2), (1^4,3), (1,1,2,3).$$

Number of complete partitions of n sequence is OEIS A126796.

Generalizing complete partitions

λ complete if and only if $\lambda_{i+1} \leq 1 + \sum_{j=0}^{i} \lambda_i$.

The complete partitions of 7 are

$(1^7), (1^5, 2), (1, 1, 1, 2, 2), (1, 2, 2, 2), (1^4, 3), (1, 1, 2, 3), (1, 1, 1, 4), (1, 2, 4)$

λ k-tight if and only if $\lambda_{i+1} \leq k + \sum_{j=0}^{i} \lambda_i$.

2-tight partitions of 6 are

$(1^6), (1^4, 2), (1, 1, 2, 2), (2, 2, 2), (1, 1, 1, 3), (1, 2, 3), (1, 1, 4), (2, 4)$

Integer sequence is same as 1-tight (complete) partitions, offset 1.

Generalizing complete partitions

3-tight partitions of 6 are the 2-tight

$$(1^6), (1^4, 2), (1, 1, 2, 2), (2, 2, 2), (1, 1, 1, 3), (1, 2, 3), (1, 1, 4), (2, 4)$$

along with $(3, 3)$. In other words, all but $(1, 5)$ and (6).

As tightness goes up, the restriction loosens: All partitions of n are n-tight (and $(n-1)$-tight). Let $a_k(n)$ be the number of k-tight partitions of n.

Sequences: 3-tight OEIS A286929, 4-tight OEIS A286097 (without that terminology, so far).

Broadening number theory

Classic multiplicative functions of number theory:

- Euler ϕ function $\phi(30) = 8$,
- sum-of-divisors $\sigma(30) = 72$,
- Möbius function $\mu(30) = -1$.

Robert Schneider (2017), Arithmetic of partitions and the q-bracket operator, *Proceedings of the American Mathematical Society* 145: 1953–1967

Ambitious program to extend number theory to integer partitions; "classic" is just a special case.

Möbius functions

Number theory: For $n = p_1^{a_1} \cdots p_t^{a_t}$ (allowing $t = 0$ for $n = 1$),

$$\mu(n) = \begin{cases} 0 & \text{if any } a_i \geq 2, \\ (-1)^t & \text{if all } a_i = 1. \end{cases}$$

Highlights square-free numbers by parity of (distinct) prime divisors. E.g., $\mu(30) = -1$.

Partition theory: For $\lambda = (\lambda_1, \ldots, \lambda_t)$,

$$\mu(\lambda) = \begin{cases} 0 & \text{if any } \lambda_i = \lambda_{i+1}, \\ (-1)^t & \text{if parts distinct.} \end{cases}$$

Highlights distinct partitions by parity of length. E.g., $\mu(1, 1, 4) = 0$, $\mu(2, 4) = 1$, $\mu(1, 2, 3) = -1$.

Amazing connection #1, matrix version

Build a square matrix from $-\mu$ applied to partitions of n separated by largest part. E.g.,

$(1,3,4)$	$(1,2,5)$			(8)
$(1^4,4)$, $(1,1,2,4)$, $(2,2,4)$, $(4,4)$	$(1,1,1,5)$	$(1,1,6)$		
	$(3,5)$	$(2,6)$	$(1,7)$	

Amazing connection #1, matrix version

Build a square matrix from $-\mu$ applied to partitions of *n* separated by largest part. E.g.,

$(1,3,4)$	$(1,2,5)$			(8)
$(1^4,4)$, $(1,1,2,4)$, $(2,2,4)$, $(4,4)$	$(1,1,1,5)$	$(1,1,6)$		
	$(3,5)$	$(2,6)$	$(1,7)$	
1	0	-1	-1	1

Amazing connection #1, matrix version

Let $\mu_{i,j} = \sum -\mu(\lambda)$ over $\lambda \vdash (i+1)$ with largest part $j+1$.

$$\mu_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 & -1 & 1 \end{pmatrix}.$$

Lower triangular by construction, colored values from previous slide.

Amazing connection #1, matrix version

Write c_n for the sequence of number of complete partitions of i for $0 \le i \le n-1$.

$$\mu_9 \cdot c_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 4 \\ 5 \\ 8 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}!$$

Amazing connection #1, bijection

$$\begin{pmatrix} 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 4 \\ 5 \\ 8 \\ 10 \end{pmatrix} = 1 - 2 - 4 + 5 = 0$$

can be shown from a bijection between
[odd length distinct partitions of 8 $\times$ some complete partitions] &
[even length distinct partitions of 8 $\times$ some complete partitions] .

Distinct partitions of n with largest part k "dot with" complete partitions of $k-2$:

$(1,3,4)$			$(1,1)$
$(1,2,5)$	$(5,3)$		$(1,1,1)$, $(2,1)$
	$(6,2)$	$\times$	(1^4), $(1,1,2)$
	$(7,1)$		(1^5), $(1,1,1,2)$, $(1,2,2)$, $(1,1,3)$
(8)			(1^6), $(1^4,2)$, $(1,1,2,2)$, $(1,1,1,3)$, $(1,2,3)$

$$1+2\cdot 2+2+4+5=16 \text{ elements}$$

Amazing connection #1, bijection

$$
\begin{array}{rlcrl}
[(1,3,4), & (1,1)] & \longleftrightarrow & [(1,7), & (1,1,3)] \\
[(3,5), & (1,2)] & \longleftrightarrow & [(8), & (1,2,3)] \\
[(3,5), & (1,1,1)] & \longleftrightarrow & [(8), & (1,1,1,3)] \\
[(1,2,5), & (1,2)] & \longleftrightarrow & [(1,7), & (1,2,2)] \\
[(1,2,5), & (1,1,1)] & \longleftrightarrow & [(1,7), & (1,1,1,2)] \\
[(2,6), & (1,1,2)] & \longleftrightarrow & [(8), & (1,1,2,2)] \\
[(1,7), & (1^5)] & \longleftrightarrow & [(8), & (1^5,1)] \\
[(2,6), & (1^4)] & \longleftrightarrow & [(8), & (1^4,2)]
\end{array}
$$

Involution changes parity of length for distinct partitions.

Amazing connection #1, bijection

$$
\begin{array}{rlcrl}
[\,(1,3,4), & (1,1)\,] & \longleftrightarrow & [\,(1,7), & (1,1,3)\,] \\
[\,(3,5), & (1,2)\,] & \longleftrightarrow & [\,(8), & (1,2,3)\,] \\
[\,(3,5), & (1,1,1)\,] & \longleftrightarrow & [\,(8), & (1,1,1,3)\,] \\
[\,(1,2,5), & (1,2)\,] & \longleftrightarrow & [\,(1,7), & (1,2,2)\,] \\
[\,(1,2,5), & (1,1,1)\,] & \longleftrightarrow & [\,(1,7), & (1,1,1,2)\,] \\
[\,(2,6), & (1,1,2)\,] & \longleftrightarrow & [\,(8), & (1,1,2,2)\,] \\
[\,(1,7), & (1^5)\,] & \longleftrightarrow & [\,(8), & (1^5,1)\,] \\
[\,(2,6), & (1^4)\,] & \longleftrightarrow & [\,(8), & (1^4,2)\,]
\end{array}
$$

Involution changes parity of length for distinct partitions. When two largest parts over both partitions are in the distinct partition, add them in distinct partition, suffix second largest part in complete partition. For reverse, difference and second largest in distinct partition, remove second largest from complete partition.

Amazing connection #2, matrix version

Form a matrix from the number of k-tight partitions of n. Specifically, $\gamma_{i,j} = a_j(i+j-2) = \#(j\text{-tight partitions of } i+j-2)$.

$$\gamma_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 4 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 5 & 5 & 4 & 3 & 2 & 1 & 1 & 0 & 0 \\ 8 & 8 & 6 & 5 & 3 & 2 & 1 & 1 & 0 \\ 10 & 10 & 9 & 6 & 5 & 3 & 2 & 1 & 1 \end{pmatrix}.$$

Lower triangular by construction, first column is c_9, second column matches first by connection of 1-tight and 2-tight partitions, diagonals "stabilize" to partition numbers.

Amazing connection #2, side note

Connection between complete partitions of n with largest part k (counted by $c(n,k)$) and k-tight partitions.

$c(n,k) = a_{k+1}(n-k) - a_{k+2}(n-k-1)$

Bijection $A_{k+1}(n-k) \longleftrightarrow C(n,k) \cup A_{k+2}(n-k-1)$.

Suppose $\lambda \in A_{k+1}(n-k)$, so that $\lambda \vdash n-k$ and $\lambda_j \leq k+1+\sum_{i=0}^{j-1} \lambda_i$ for each part λ_j.

If $\lambda_1 = 1$, then let $\lambda' = (\lambda_2, \ldots, \lambda_t) \vdash n-k-1$. Note that

$$\lambda'_1 = \lambda_2 \leq k+1+\lambda_1 = k+2$$

and the $(k+2)$-tight condition holds for the subsequent parts, so $\lambda' \in A_{k+2}(n-k-1)$.

Bijection continued

If $\lambda_1 \neq 1$, then let λ^* be λ with k additional parts 1, so $\lambda^* \vdash n$. We need to show that λ^* is complete, i.e., that $\lambda^*_j \leq k + 1 + \sum_{i=0}^{j-1} \lambda^*_i$ for each part λ^*_j. This clearly holds for the first k parts of λ^*, which are each 1. Next,

$$\lambda^*_{k+1} = \lambda_1 \leq k + 1 + \lambda_0 = k + 1$$

so that $\lambda^*_{k+1} \leq 1 + \sum_{i=0}^{k} \lambda^*_i = k + 1$ as desired. The 1-tight condition holds for subsequent parts, so $\lambda^* \in C(n, k)$.

So the γ matrix could be defined from complete partitions separated by largest part.

Amazing connection #2, matrix version

$$\mu_9 \cdot \gamma_9 = Id_9$$

Equivalently,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 4 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 5 & 5 & 4 & 3 & 2 & 1 & 1 & 0 & 0 \\ 8 & 8 & 6 & 5 & 3 & 2 & 1 & 1 & 0 \\ 10 & 10 & 9 & 6 & 5 & 3 & 2 & 1 & 1 \end{pmatrix}.$$

Amazing connection #2, generating function argument

Lemma: Given a positive integer k and a partition λ, there is a maximal k-tight subpartition λ^k of λ. Also, if $\lambda^k \vdash n$, then none of $n+1, \ldots, n+k$ can be a part of λ.

Given λ, construct λ^k by starting at λ_1 and including all parts sequentially that satisfy the definition of a k-tight partition.

Suppose that the resulting $\lambda^k \vdash n$. If any of $n+1, \ldots, n+k$ were a part of λ, then they could be included in λ^k to make a longer k-tight subpartition of λ, contradicting the maximality of λ^k.

Example: The partition $\lambda = (2,4,9)$ has $\lambda^1 = \varnothing$, $\lambda^2 = (2,4)$, and $\lambda^k = \lambda$ for $k \geq 3$.

Amazing connection #2, generating function argument

Theorem: For each positive integer k,

$$1 = \sum_{n=0}^{\infty} a_k(n)(1-q)(1-q^2)\cdots(1-q^{n+k}).$$

Rewrite the desired identity as

$$\frac{1}{\prod_{n=1}^{\infty}(1-q^n)} = \sum_{n=0}^{\infty} \frac{a_k(n)q^n}{(1-q^{n+k+1})(1-q^{n+k+2})\cdots}$$

or

$$\sum_{n=0}^{\infty} p(n)q^n = \sum_{n=0}^{\infty} \frac{\sum_{\pi \in A_k(n)} q^{|\pi|}}{\prod_{j=n+k+1}^{\infty}(1-q^j)}. \qquad (1)$$

Idea: Count partitions by their maximal k-tight subpartition.

Amazing connection #2, generating function argument

Proof, continued.

By the lemma, every partition λ has a maximal k-tight subpartition λ^k. Also, if $\lambda^k \vdash n$, then none of $n+1, \ldots, n+k$ are parts of λ. However, there is no constraint of the parts of λ greater than $n+k$, since the absence of $n+1, \ldots, n+k$ in λ means that no larger k-tight subpartition can be produced. Therefore,

$$\sum_{n=0}^{\infty} \frac{\sum_{\pi \in A_k(n)} q^{|\pi|}}{\prod_{j=n+k+1}^{\infty}(1-q^j)} = \sum_{n=0}^{\infty} \frac{a_k(n) q^n}{(1-q^{n+k+1})(1-q^{n+k+2})\cdots}$$

generates all partitions whose maximal k-tight subpartition is a partition of n. Summing over all $n \geq 0$ gives (1) and consequently the desired result.

To-do list

- I believe the Amazing Connection #1 bijection generalizes from complete partitions to k-tight partitions, i.e., can extend to a bijective proof of Amazing Connection #2.
- Connect the generating function theorem more explicitly to Amazing Connection #2.
- Deeper understanding of the connection between "signed" distinct partitions and complete partitions.