Counting in Austrian Solitaire

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Mathematics Seminar
Mahidol University International College
15 May 2019

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#10!

Overview

Outline:

- Bulgarian Solitaire
  - History
  - Cycle states
  - Garden of Eden states
- Austrian Solitaire
  - History
  - Counting total states & Garden of Eden states*
  - Cycle states
- Open questions

* New results are joint work with James Sellers (Pennsylvania State University/University of Minnesota – Duluth) and Robson da Silva (Federal University of São Paulo, Brazil).

Train Talk

"Oh, you're a mathematician! Let me show you something interesting."

But I'm trying to work on my talk, I thought. As the train sped along, the man sitting across from me looked eager. *OK, let's get this over with.*

"Here are fifteen playing cards. Arrange them into piles; as many piles as you like, each with as many cards as you like."

I made five piles with heights 3, 1, 4, 1, 6. The π reference went unnoticed.

"Now take one card from each pile to make a new pile."

The operation left me with piles of $3 - 1 = 2$ cards, $4 - 1 = 3$ cards, $6 - 1 = 5$ cards, two empty piles from $1 - 1 = 0$, and a new pile of 5 cards. I realized that the order of the piles does not matter, so for consistency I put them in non-increasing order, 5, 5, 3, 2.

"Now do it again and again. I know what will happen!" He looked away.

Did he already think through the iterations? How long will this go? I was curious now. Here is the sequence of pile sizes:

$$\begin{array}{ccccccc} (6,4,3,1,1) & \to & (5,5,3,2) & \to & (4,4,4,2,1) & \to & (5,3,3,3,1) \\ & \to & (5,4,2,2,2) & \to & (5,4,3,1,1,1) & \to & (6,4,3,2) \end{array}$$

Oh no, that's almost where I started. When will this end? But then, suddenly, it did end:

$$(6,4,3,2) \quad \to \quad (5,4,3,2,1) \quad \to \quad (5,4,3,2,1) \text{ again.}$$

"Hmm," I said.

"You ended with one pile of 5 cards, one of 4, one of 3, one of 2, and one of 1, didn't you?" He looked at the cards. "Yes! That's always what happens. Try again!"

I started with a single pile of 15 cards. It took more moves, but I did indeed end up with the 5, 4, 3, 2, 1 pattern. Then I started with three piles of 5 cards. It took even more moves, but ended at the same fixed point.

Three examples is not a proof, but the claim was now reasonable. *Why would it always go there? How long could it take to reach this fixed point? What happens with other numbers of cards?*

"This *is* interesting," I admitted.

Mathematical setting

A partition of n is an unordered collection of positive integers whose sum is n.

$$P(4) = \{4, 31, 22, 211, 1111\}; \qquad p(4) = 5$$

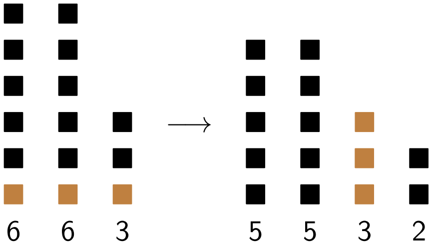
Write $\lambda \in P(n)$ (equivalently, $\lambda \vdash n$) as $\lambda = (\lambda_1, \ldots, \lambda_t)$. The λ_i are the parts, given in nonincreasing order. The length of λ is the number of parts t.

The operation induces a finite dynamical system on $P(n)$, example of "partition dynamics."

The operation on partitions

$(\lambda_1, \ldots, \lambda_t) \to (t, \lambda_1 - 1, \ldots, \lambda_t - 1)$, presented in nonincreasing order so that reordering may be necessary.

Can visualize with Ferrers diagrams of partitions:



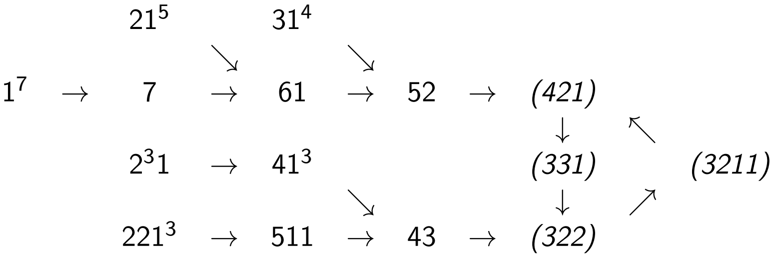
(You may recall that $(6, 4, 3, 1, 1)$ also maps to $(5, 5, 3, 2)$ . . .)

Operation on partitions of 6

$$
\begin{array}{ccccccccc}
 & & & & 21^4 & & & & \\
 & & & & & \searrow & & & \\
 & & 1^6 & \to & 6 & \to & 51 & \to & 42 \to \mathit{(321)} \\
 & & & & & & & \nearrow & \\
2211 \to 411 & \to & 33 & \to & 2^3 & \to & 31^3 & &
\end{array}
$$

Here too everything leads to one state, $(3, 2, 1)$.

Operation on partitions of 7



Here everything leads to a cycle of 4 partitions rather than 1.

Operation on partitions of 8

$$
\begin{array}{ccccccccccccc}
 & 21^6 & & 31^5 & & 321^3 & & & & & & & \\
 & & \searrow & & \searrow & & \searrow & & & & & & \\
1^8 \to 8 & & \to & 71 & \to & 62 & \to & 521 & \to & \mathit{(431)} & \leftarrow & \mathit{(4211)} \\
 & & & & & & & & & \downarrow & & \uparrow \\
 & & & 2^311 & \to & 51^3 & \to & 44 & \to & \mathit{(332)} & \to & \mathit{(3221)} \\
\hline
 & & & 2^4 & \to & 41^4 & \to & 53 & \to & \mathit{(422)} & \leftrightarrow & \mathit{(3311)} \\
 & & & & & & \nearrow & & & & & \\
 & & & 221^4 & \to & 611 & & & & & &
\end{array}
$$

Here two components, into a 4-cycle or 2-cycle.

History

All 1980/1981. Journal (language), object moved, poser, solver.

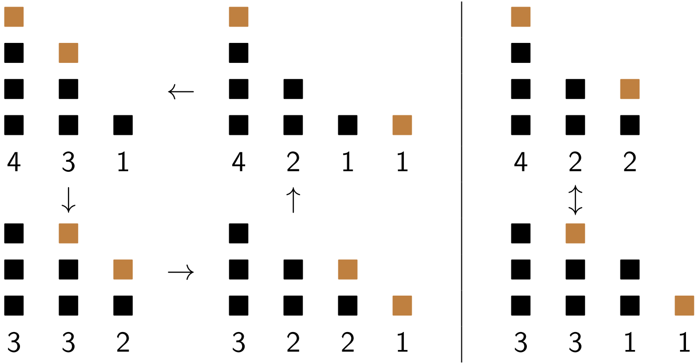
- *Kvant* (Russian), books, Gutenmacher, Toom
- *Obuchenieto po matematika* (Bulgarian), balls, Petkov, Bojanov
- *Elementa* (Swedish), cards, Henrik Eriksson both

J. Brandt, Cycles of partitions, *Proc. Amer. Math. Soc.* 85 (1982) 483–486

M. Gardner, Tasks you cannot help finishing no matter how hard you try to block finishing them, *Sci. Amer.* 249 (1983) 12–21

H., 30 years of Bulgarian Solitaire, *College Math. J.* 43 (2012) 135–140

Bulgarian Solitaire cycle partitions



Bulgarian Solitaire cycle partitions

Toom, Bojanov, Eriksson, Brandt

Write $n = T_m + m'$, the mth triangle number plus m', $0 \leq m' \leq m$. The cyclic partitions of n under Bulgarian Solitaire are

$$(m + \delta_m, m - 1 + \delta_{m-1}, \ldots, 2 + \delta_2, 1 + \delta_1, \delta_0)$$

where each δ_i is 0 or 1 with exactly m' of them 1. There are $\binom{m+1}{m'}$ cyclic partitions.

So if n is a triangular number, i.e., if $m' = 0$, then all the $\delta_i = 0$ and the unique cyclic partition is $(m, m-1, \ldots, 2, 1)$. For $m' \geq 1$, each $\delta_i = 1$ corresponds to a dot in the diagonal above the triangular partition.

Bulgarian Solitaire components

Brandt

Under Bulgarian Solitaire, $P(n)$ has

$$\frac{1}{m+1} \sum_{d|(m+1,m')} \phi(d) \binom{(m+1)/d}{m'/d}$$

cycles, which is also the number of connected components, where ϕ is the Euler function.

Uses Pólya enumeration to group partitions in cycles, e.g.,

$$(1,1,0,0) \to (0,1,1,0) \to (0,0,1,1) \to (1,0,0,1) \to (1,1,0,0)$$

along the diagonal above $(3,2,1)$ in the 4-cycle of $P(8)$.

Bulgarian Solitaire on partitions of 6

The discrete dynamical system distinguishes two types of state: sinks and sources. Following cellular automata, call partitions with no predecessor under the operation Garden of Eden states.

$\boxed{21^4} \searrow 51$

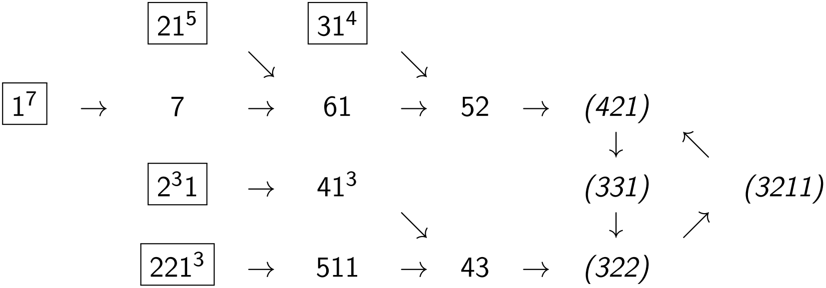
$\boxed{1^6} \to 6 \to 51 \to 42 \to$ *(321)*

$\boxed{2211} \to 411 \to 33 \to 2^3 \to 31^3 \nearrow 42$

1 component, 1 *(fixed point)* 321.

3 Garden of Eden states $2211, 21^4, 1^6$.

Bulgarian Solitaire on partitions of 7



1 component, 4 *(cyclic partitions)* 421, 331, 322, 3211.

5 GE's $31^4, 2^31, 221^3, 21^5, 1^7$.

Bulgarian Solitaire on partitions of 8

$$\begin{array}{ccccccccccc}
 & \boxed{21^6} & \searrow & \boxed{31^5} & \searrow & \boxed{321^3} & \searrow & & & & \\
\boxed{1^8} & \to 8 & \to & 71 & \to & 62 & \to & 521 & \to & (431) \leftarrow & (4211) \\
 & & & & & & & & & \downarrow & \uparrow \\
 & & & \boxed{2^311} & \to & 51^3 & \to & 44 & \to & (332) \to & (3221)
\end{array}$$

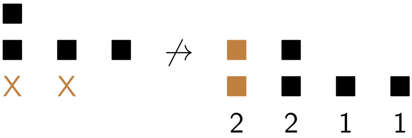
$$\begin{array}{ccccccc}
\boxed{2^4} & \to & 41^4 & \to & 53 & \to & (422) \leftrightarrow (3311) \\
 & & & \nearrow & & & \\
\boxed{221^4} & \to & 611 & & & &
\end{array}$$

2 components (15+7), 6 *(cyclic)* 431, 4211, 332, 3221; 422, 3311.

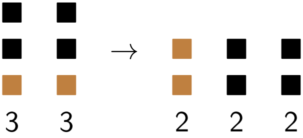
7 $\boxed{\text{GE's}}$ $321^3, 31^5, 2^311, 21^6, 1^8; 2^4, 221^4$.

Bulgarian Solitaire Garden of Eden partitions

Why is 2211 a GE-partition?



Whereas, 2^3 does have a pre-image.



Bulgarian Solitaire Garden of Eden partitions

The rank of a partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ is $\lambda_1 - t$. Defined by Freeman Dyson in 1944 as a potential explanation for a pattern in $p(n)$ values found by Ramanujan, eventually proved by Atkin & Swinnerton-Dyer.

H. & Jones, *Electr. J. Combin* 2006; H. & Sellers, *Integers* 2007

$\lambda \in P(n)$ is a GE-partition for Bulgarian solitaire if and only if $\text{rank}(\lambda) \leq -2$. The number of them is

$$p(n-3) - p(n-9) + p(n-18) - p(n-30) + \cdots$$

where $3, 9, 18, 30, \ldots$ are $3T_m$.

We gave two proofs of the partition formula, one using Dyson's generating functions related to rank, one combinatorial using an adjoint mapping Dyson defined in 1969.

$$\left(\prod_{i=1}^{\infty} \frac{1}{1-q^i}\right) \sum_{r=1}^{\infty} (-1)^{r-1} \left(q^{(3r^2-r)/2} - q^{(3r^2+r)/2}\right) q^{mr}$$

describes the number of partitions of n with rank m. Writing $ge(n)$ for the number of GE-partitions of n,

$$\sum_{n=0}^{\infty} ge(n) q^n = \sum_{m=2}^{\infty} \left(\prod_{i=1}^{\infty} \frac{1}{1-q^i}\right) \sum_{r=1}^{\infty} (-1)^{r-1} \left(q^{(3r^2-r)/2} - q^{(3r^2+r)/2}\right) q^{mr}$$

$$= \cdots = \left(\sum_{n=0}^{\infty} p(n) q^n\right) \sum_{r=1}^{\infty} (-1)^{r-1} q^{3\binom{r}{2}}$$

Three volume *Capital and Interest* by Eugen von Böhm-Bawerk (1884, 1889, 1921), emphasizes the means and duration of production. Led to 1930s debate between John Maynard Keynes and Friedrich Hayek:

> *Hayek accused Keynes of insufficient attention to the nature of capital in production. (By "capital" I mean the physical production structure of the economy, including machinery, buildings, raw materials, and human capital – skills).* —*Austrian Capital Theory, Peter Lewin, 2012*

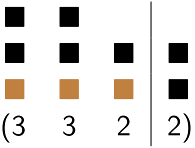
For us, we will consider the depreciation of machine value over time and saving money to buy new machines.

Austrian Solitaire as an application

A 2-parameter variation on Bulgarian Solitaire: In addition to the parameter n (the "number of dots"), there is a machine cost $L \leq n$.

A state of Austrian Solitaire consists of a partition $\lambda \vdash m \leq n$ with each part at most L and a "bank" $b < L$ with $m + b = n$. Write $(\lambda|b)$.

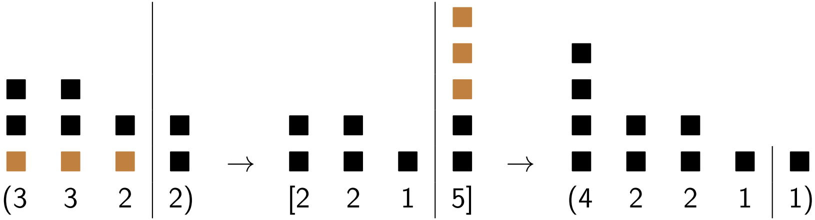
The operation begins the same way: Take one dot from each pile (but not the bank). Think of each pile as the remaining value of a production machine and the operation corresponds to the passage of time and the depreciation of each machine.



Austrian Solitaire as an application

As a good Austrian capitalist, you put aside money equivalent to the total depreciation of the system into a sinking fund for buying new machines (that cost L each). So the next step of the Austrian Solitaire operation takes the collected dots plus the bank and buys as many machines as possible. That is, add as many parts L as you can. Any remaining money is left in the bank.

Example with $n = 10$ and $L = 4$ from $(332|2)$.



Austrian Solitaire abstractly

Akin & Davis: "The economic interpretation shows that anything can become applied mathematics"

Let $P(n, L)$ denote the partitions of n where each part is at most L. The generating function for $p(n, L) = |P(n, L)|$ is

$$\sum_{n=0}^{\infty} p(n, L) q^n = \prod_{i=1}^{L} \frac{1}{1 - q^i}$$

and direct formulas can be derived for small L.

A state $(\lambda|b)$ of Austrian solitaire satisfies $\lambda \in P(n - b, L)$. The defining operation for Austrian Solitaire is

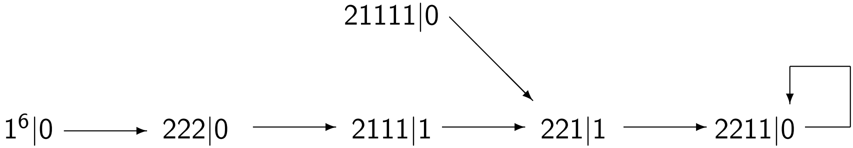
$$A((\lambda|b)) = (L^{\lfloor \frac{t+b}{L} \rfloor}, \lambda_1 - 1, \ldots, \lambda_t - 1 \mid (b + t) \bmod L)$$

where $\lfloor \cdot \rfloor$ denotes the least integer function.

Austrian Solitaire system examples

Given n and L, the Austrian Solitaire system consists of all states $(\lambda|0)$ for $\lambda \in P(n, L)$ and all iterated images under the operation A.

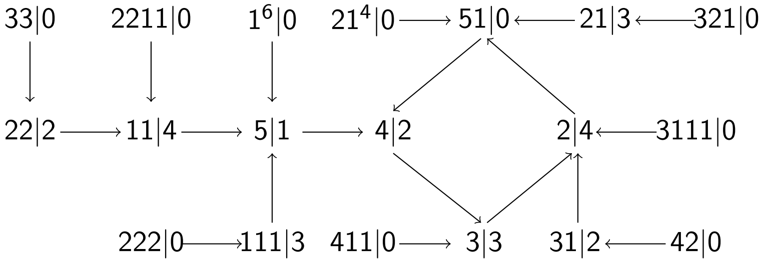
The $n = 6$, $L = 2$ system:



Note that $(1^5|1)$ is not in the system...

Austrian Solitaire system examples

The $n = 6$, $L = 5$ system:



$P(6, 5) = 10$ (all but (6)) then 9 with positive bank.

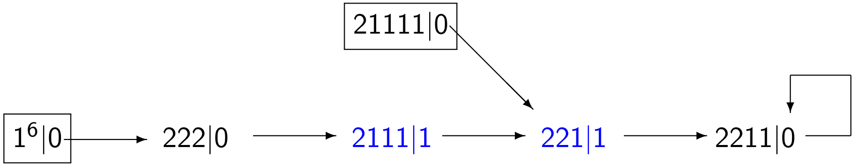
Austrian Solitaire types of states

So Austrian Solitaire has three types of distinguished states: cycle, Garden of Eden, and positive bank states (needed to determine the size of a given system).

Austrian Solitaire types of states

So Austrian Solitaire has three types of distinguished states: cycle, Garden of Eden, and positive bank states (needed to determine the size of a given system).

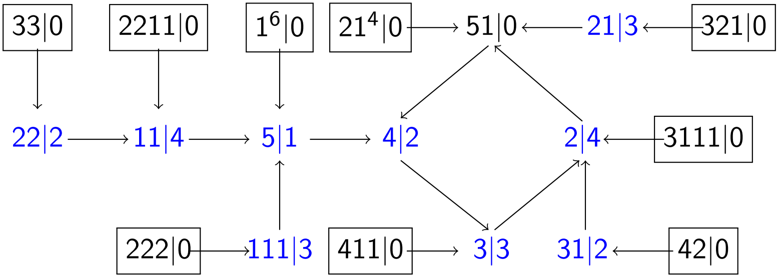
But it seems two of these are connected:



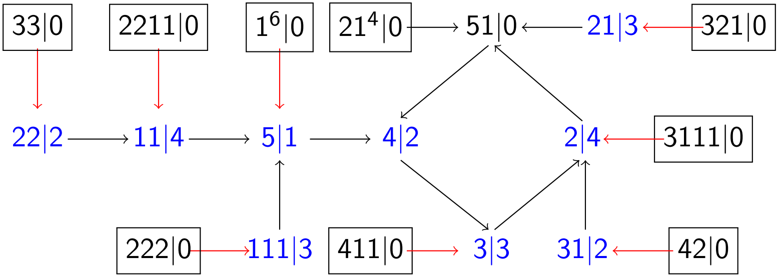
The $n = 6$, $L = 2$ system has 2 GEs and 2 positive bank states.

Austrian Solitaire types of states

The $n = 6$, $L = 5$ system has 9 of each.



Connection GE states to positive bank states?



Path from each GE to a positive bank state? Uniqueness? Length?

Connecting GE and positive bank states

A helpful statistic on $\lambda = (L^a, \lambda_{a+1}, \ldots, \lambda_t)$ where $\lambda_{a+1} < L$ is

$$h(\lambda) = \lambda_{a+1} + 1 + \cdots + \lambda_t + 1 = \lambda_{a+1} + \cdots + \lambda_t + t - a.$$

Proposition 1: Valid positive bank states

Suppose positive integers n, L, and z with $1 \leq z \leq L - 1$ are given, and $\mu \in P(n - z, L)$. Then $(\mu|z)$ is a valid state for Austrian Solitaire if and only if $h(\mu) \leq n$.

For a positive bank state to be valid, it needs to have a pre-image under A (the system starts from $P(n, L)$). The statistic $h(\mu)$ sums the parts required in the preimage state. E.g., the $(332|2)$ example with $n = 10$, $L = 4$ has $h(332) = 4 + 4 + 3 = 11$, not a valid state.

Connecting GE and positive bank states

Write $\mu = (L^a, \mu a + 1, \ldots, \mu)$ where $\mu_{a+1} < L$. When $(\mu|z)$ satisfies $h(\mu) \leq n$, we can construct $\lambda \in P(n, L)$ such that $A((\lambda|0)) = (\mu|z)$:

$$\lambda = (\mu_{a+1} + 1, \ldots, \mu_t + 1, 1^{a(L+1)-t+z})$$

where $h(\mu) \leq n$ and the number of 1s is nonnegative. □

This actually shows that every positive bank state has a bank 0 preimage. So only bank 0 states can be Garden of Eden states.

Connecting GE and positive bank states

Proposition 2: At most one preimage per bank size

Suppose n and L are positive integers, z satisfies $0 \le z \le L - 1$, and $\mu \in P(n - z, L)$ such that $(\mu|z)$ is a valid state of Austrian Solitaire. For each y satisfying $0 \le y \le L - 1$, there is at most one valid state $(\lambda|y)$ with $A((\lambda|y)) = (\mu|z)$. Further, if $(\mu|z)$ has any preimage, then it has one with $y = 0$.

E.g., for $n = 6$, $L = 5$, $(2|4)$ has preimages $(3|3)$, $(31|2)$, and $(3111|0)$, nothing with bank 1 or 4.

Similar proof: Given y, the preimage bank size, explicitly construct $(\lambda|y)$, so it is unique if it exists.

For the statement about a bank 0 preimage, construct from $(\lambda|y)$ with $y \ge 1$ and $A((\lambda|y)) = (\mu|z)$ an explicit $\kappa \in P(n, L)$ with $A((\kappa|0)) = (\mu|z)$.

Connecting GE and positive bank states

In $n = 6$, $L = 5$, $(2|4)$ has preimages $(3|3)$, $(31|2)$, and $(3111|0)$.

In practice, two things can block constructing a $(\mu|z)$ preimage with a specified bank size y.

- The specified number of 1s in λ could be negative: $(2|4)$ has no bank 4 preimage because

$$\#1s = a(L+1) - t + z - y = -1.$$

- If $y \geq 1$, then λ must also satisfy the condition given in Proposition 1: $(2|4)$ has no bank 1 preimage because the prescribed $\lambda = (3,1,1)$ has $h((3,1,1)) = 8 > 6$.

Connecting GE and positive bank states

Theorem 3: A bijection

Given positive integers n and L, the number of Garden of Eden Austrian Solitaire states equals the number of positive bank Austrian Solitaire states.

Start with a Garden of Eden state and apply A until you reach a positive bank state; associate those two. Chain can be arbitrarily long (e.g., there's a length 3 connection in $n = 65$, $L = 5$).

If the path does not hit a positive bank state, then forms a bank 0 cycle (since $P(n, L) < \infty$) making some state have two bank 0 preimages, contradicting Proposition 2.

Essentially the propositions show A is an invertible function between a GE state and its associated positive bank state.

Two questions reduced to one

Write $ge(n, L)$ for the number of Garden of Eden states in the Austrian Solitaire system for parameters n and L. We've shown that the size of the system is $P(n, L) + ge(n, L)$.

GE state characterization

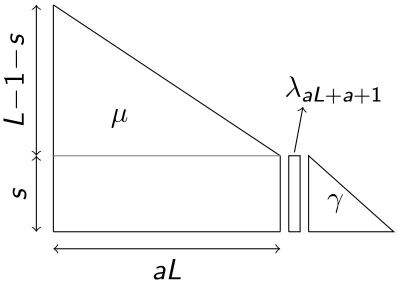
Given positive integers n and L, the Garden of Eden states for Austrian Solitaire are the $(\lambda|0)$ with $h(\lambda) > n$.

If positive bank, there's a preimage. If $h(\lambda) \leq n$, there's a preimage. □

The remaining task is to count these.

Counting GE states

$\lambda = (L^a, \lambda_{a+1}, \ldots, \lambda_t)$ and $h(\lambda) > n$ imply $t - a > aL$: there are more than aL parts of λ less than L. Let $\lambda_{aL+a+1} = s$; here is $(\lambda_{a+1}, \ldots, \lambda_t)$.

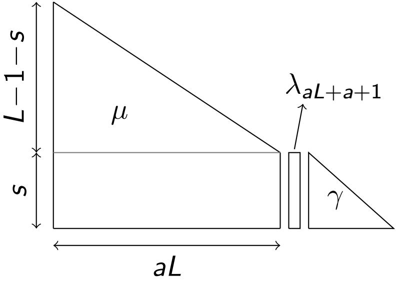


μ fits in an $(aL) \times (L - 1 - s)$ box; each part of γ is at most s.

Counting GE states

q-binomial coefficient $\begin{bmatrix} n+m \\ m \end{bmatrix}$ counts partitions in an $n \times m$ box.

$$\sum_{n=0}^{\infty} ge(n,L)q^n = \sum_{a=0}^{\infty} q^{aL} \sum_{s=1}^{L-1} q^{s(aL+1)} \begin{bmatrix} aL+L-1-s \\ aL \end{bmatrix} \frac{1}{(q;q)_s}$$



Counting GE states

$$\sum_{n=0}^{\infty} ge(n,L)q^n = \sum_{a=0}^{\infty} q^{aL} \sum_{s=1}^{L-1} q^{s(aL+1)} \begin{bmatrix} L(a+1)-s-1 \\ aL \end{bmatrix} \frac{1}{(q;q)_s}$$

$$= \cdots = \frac{1}{(q;q)_\infty} \sum_{s=1}^{L-1} \frac{(q^{L-s};q)_\infty}{(q;q)_s} q^s \sum_{a=0}^{\infty} \frac{(q^{aL+1};q)_\infty}{(q^{L(a+1)-s};q)_\infty} q^{aL(s+1)}$$

$$= \cdots = \sum_{s=1}^{L-1} \sum_{m=0}^{L-1-s} \frac{(-1)^m q^{m+s+\binom{m}{2}}}{(q;q)_s (q;q)_m (q;q)_{L-s-m-1} (1-q^{L(m+s+1)})}$$

$$= \cdots = \sum_{k=1}^{L-1} \frac{(-1)^{k-1} q^{\binom{k+1}{2}}}{(q;q)_k (q;q)_{L-1-k} (1-q^{L(k+1)})}.$$

Open questions

For Austrian Solitaire:

- Characterize the cycle states. A few partial results in Kapil Bastola's 2012 undergraduate honors thesis (guess the advisor), Proposition 2 here should be helpful. (Proposition 1 in his thesis, Theorem 3 was a conjecture.)
- Akin & Davis conjectured that an Austrian Solitaire system always has just one component (unlike Bulgarian Solitaire for $n = 8$), i.e., one just cycle. Intuition: Always adding parts L means no reordering of parts, simpler setting. Observation: In a cycle, each state has a different size bank.
- Distances from GE states to positive bank states and to cycle states.

Open questions

For Bulgarian Solitaire:

- The system has multiple components whenever n is 2 or more away from the nearest triangular number. Can we say anything about component sizes? Even the largest component? How can we identify which component includes a given partition?
- How far are GE partitions from cycles? Know minimum distance is 3. For $n = T_m$, the partition

$$\gamma_m = (m-1, m-1, m-2, m-3, \ldots, 3, 2, 1, 1) \vdash n$$

  is at maximal distance $m(m-1)$ from the fixed point. Griggs & Ho 1998 conjectured maximum lengths for other n.
- How many partitions at the maximal distance from the cycle/fixed point? Whole spectrum by distance.

Open questions

For Bulgarian Solitaire:

- What is the interaction between conjugation of partitions and Bulgarian Solitaire? For $n = T_m$, Bentz 1987 showed that the path from γ_m to the fixed point consists of nested conjugate pairs, e.g.,

$$2211 \to 411 \to 33 \to 2^3 \to 31^3 \to 42 \to 321$$

And similar questions for related operations; partition dynamics!