Counting in Austrian Solitaire

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Mathematics Seminar
Mahidol University International College
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#10!
Outline:

- **Bulgarian Solitaire**
  - History
  - Cycle states
  - Garden of Eden states

- **Austrian Solitaire**
  - History
  - Counting total states & Garden of Eden states*
  - Cycle states

- **Open questions**

* New results are joint work with James Sellers (Pennsylvania State University/University of Minnesota – Duluth) and Robson da Silva (Federal University of São Paulo, Brazil).
“Oh, you’re a mathematician! Let me show you something interesting.”

*But I’m trying to work on my talk*, I thought. As the train sped along, the man sitting across from me looked eager. *OK, let’s get this over with.*

“Here are fifteen playing cards. Arrange them into piles; as many piles as you like, each with as many cards as you like.”

I made five piles with heights 3, 1, 4, 1, 6. The \(\pi\) reference went unnoticed.

“Now take one card from each pile to make a new pile.”

The operation left me with piles of \(3 - 1 = 2\) cards, \(4 - 1 = 3\) cards, \(6 - 1 = 5\) cards, two empty piles from \(1 - 1 = 0\), and a new pile of 5 cards. I realized that the order of the piles does not matter, so for consistency I put them in non-increasing order, 5, 5, 3, 2.

“Now do it again and again. I know what will happen!” He looked away.

*Did he already think through the iterations? How long will this go?* I was curious now. Here is the sequence of pile sizes:
(6, 4, 3, 1, 1) → (5, 5, 3, 2) → (4, 4, 4, 2, 1) → (5, 3, 3, 3, 1)
→ (5, 4, 2, 2, 2) → (5, 4, 3, 1, 1, 1) → (6, 4, 3, 2)

Oh no, that’s almost where I started. When will this end? But then, suddenly, it did end:

(6, 4, 3, 2) → (5, 4, 3, 2, 1) → (5, 4, 3, 2, 1) again.

“Hmm,” I said.

“You ended with one pile of 5 cards, one of 4, one of 3, one of 2, and one of 1, didn’t you?” He looked at the cards. “Yes! That’s always what happens. Try again!”

I started with a single pile of 15 cards. It took more moves, but I did indeed end up with the 5, 4, 3, 2, 1 pattern. Then I started with three piles of 5 cards. It took even more moves, but ended at the same fixed point.

Three examples is not a proof, but the claim was now reasonable. Why would it always go there? How long could it take to reach this fixed point? What happens with other numbers of cards?

“This is interesting,” I admitted.
A partition of $n$ is an unordered collection of positive integers whose sum is $n$.

$$P(4) = \{4, 31, 22, 211, 1111\}; \quad p(4) = 5$$

Write $\lambda \in P(n)$ (equivalently, $\lambda \vdash n$) as $\lambda = (\lambda_1, \ldots, \lambda_t)$. The $\lambda_i$ are the parts, given in nonincreasing order. The length of $\lambda$ is the number of parts $t$.

The operation induces a finite dynamical system on $P(n)$, example of “partition dynamics.”
The operation on partitions

$$(\lambda_1, \ldots, \lambda_t) \rightarrow (t, \lambda_1 - 1, \ldots, \lambda_t - 1),$$ presented in nonincreasing order so that reordering may be necessary.

Can visualize with Ferrers diagrams of partitions:

\begin{align*}
\begin{array}{c}
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\end{array}
\rightarrow
\begin{array}{c}
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\end{array}
\begin{array}{c}
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\end{array}
\begin{array}{c}
6 \\
6 \\
3 \\
\end{array}
\rightarrow
\begin{array}{c}
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\end{array}
\begin{array}{c}
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\end{array}
\begin{array}{c}
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\end{array}
\begin{array}{c}
5 \\
5 \\
3 \\
2 \\
\end{array}
\end{align*}

(You may recall that $(6, 4, 3, 1, 1)$ also maps to $(5, 5, 3, 2) \ldots$)
Operation on partitions of 6

Here too everything leads to one state, $(3, 2, 1)$. 

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Austrian Solitaire
Here everything leads to a cycle of 4 partitions rather than 1.
Operation on partitions of 8

\[
\begin{array}{ccc}
21^6 & 31^5 & 321^3 \\
1^8 \rightarrow 8 & \rightarrow 71 & \rightarrow 62 & \rightarrow 521 & \rightarrow (431) \leftrightarrow (4211) \\
2^311 & \rightarrow 51^3 & \rightarrow 44 & \rightarrow (332) \rightarrow (3221) \\
2^4 & \rightarrow 41^4 & \rightarrow 53 & \rightarrow (422) \leftrightarrow (3311) \\
221^4 & \rightarrow 611 \\
\end{array}
\]

Here two components, into a 4-cycle or 2-cycle.
History


- *Kvant* (Russian), books, Gutenmacher, Toom
- *Obuchenieto po matematika* (Bulgarian), balls, Petkov, Bojanov
- *Elementa* (Swedish), cards, Henrik Eriksson both


M. Gardner, Tasks you cannot help finishing no matter how hard you try to block finishing them, *Sci. Amer.* 249 (1983) 12–21

Bulgarian Solitaire cycle partitions

4 3 1 4 2 1 1
↓ ↑
3 3 2 3 2 2 1

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Write \( n = T_m + m' \), the \( m \)th triangle number plus \( m' \), \( 0 \leq m' \leq m \). The cyclic partitions of \( n \) under Bulgarian Solitaire are

\[
(m + \delta_m, m - 1 + \delta_{m-1}, \ldots, 2 + \delta_2, 1 + \delta_1, \delta_0)
\]

where each \( \delta_i \) is 0 or 1 with exactly \( m' \) of them 1. There are \( \binom{m+1}{m'} \) cyclic partitions.

So if \( n \) is a triangular number, i.e., if \( m' = 0 \), then all the \( \delta_i = 0 \) and the unique cyclic partition is \((m, m - 1, \ldots, 2, 1)\). For \( m' \geq 1 \), each \( \delta_i = 1 \) corresponds to a dot in the diagonal above the triangular partition.
Under Bulgarian Solitaire, $P(n)$ has

$$\frac{1}{m+1} \sum_{d|(m+1,m')} \phi(d) \binom{(m+1)/d}{m'/d}$$

cycles, which is also the number of connected components, where $\phi$ is the Euler function.

Uses Pólya enumeration to group partitions in cycles, e.g.,

$$(1, 1, 0, 0) \rightarrow (0, 1, 1, 0) \rightarrow (0, 0, 1, 1) \rightarrow (1, 0, 0, 1) \rightarrow (1, 1, 0, 0)$$

along the diagonal above $(3, 2, 1)$ in the 4-cycle of $P(8)$.
The discrete dynamical system distinguishes two types of state: sinks and sources. Following cellular automata, call partitions with no predecessor under the operation Garden of Eden states.

![Graph showing Bulgarian Solitaire on partitions of 6]

1 component, 1 (fixed point) 321.
3 Garden of Eden states 2211, 21^4, 1^6.
Bulgarian Solitaire on partitions of 7

1 component, 4 \((cyclic\ partitions)\) 421, 331, 322, 3211.
5 \(GE's\) 31, 231, 221, 21, 17.
Bulgarian Solitaire on partitions of 8

2 components \((15+7)\), 6 \((cyclic)\) \(431, 4211, 332, 3221; 422, 3311\).

7 \(GE's\) \(321^3, 31^5, 2^311, 21^6, 1^8; 2^4, 221^4\).
Why is 2211 a GE-partition?

Whereas, $2^3$ does have a pre-image.
The rank of a partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ is $\lambda_1 - t$. Defined by Freeman Dyson in 1944 as a potential explanation for a pattern in $p(n)$ values found by Ramanujan, eventually proved by Atkin & Swinnerton-Dyer.


$\lambda \in P(n)$ is a GE-partition for Bulgarian solitaire if and only if $\text{rank}(\lambda) \leq -2$. The number of them is

$$p(n - 3) - p(n - 9) + p(n - 18) - p(n - 30) + \cdots$$

where $3, 9, 18, 30, \ldots$ are $3T_m$. 
We gave two proofs of the partition formula, one using Dyson’s generating functions related to rank, one combinatorial using an adjoint mapping Dyson defined in 1969.

\[
\left( \prod_{i=1}^{\infty} \frac{1}{1-q^i} \right) \sum_{r=1}^{\infty} (-1)^{r-1} \left( q^{(3r^2-r)/2} - q^{(3r^2+r)/2} \right) q^{mr}
\]

describes the number of partitions of \( n \) with rank \( m \). Writing \( ge(n) \) for the number of GE-partitions of \( n \),

\[
\sum_{n=0}^{\infty} ge(n)q^n = \sum_{m=2}^{\infty} \left( \prod_{i=1}^{\infty} \frac{1}{1-q^i} \right) \sum_{r=1}^{\infty} (-1)^{r-1} \left( q^{(3r^2-r)/2} - q^{(3r^2+r)/2} \right) q^{mr}
\]

\[
= \cdots = \left( \sum_{n=0}^{\infty} p(n)q^n \right) \sum_{r=1}^{\infty} (-1)^{r-1} q^{3(\binom{r}{2})}
\]
Three volume *Capital and Interest* by Eugen von Böhm-Bawerk (1884, 1889, 1921), emphasizes the means and duration of production. Led to 1930s debate between John Maynard Keynes and Friedrich Hayek:

*Hayek accused Keynes of insufficient attention to the nature of capital in production. (By “capital” I mean the physical production structure of the economy, including machinery, buildings, raw materials, and human capital – skills).*  —*Austrian Capital Theory*, Peter Lewin, 2012

For us, we will consider the depreciation of machine value over time and saving money to buy new machines.
Austrian Solitaire as an application

A 2-parameter variation on Bulgarian Solitaire: In addition to the parameter $n$ (the “number of dots”), there is a machine cost $L \leq n$.

A state of Austrian Solitaire consists of a partition $\lambda \vdash m \leq n$ with each part at most $L$ and a “bank” $b < L$ with $m + b = n$. Write $(\lambda|b)$.

The operation begins the same way: Take one dot from each pile (but not the bank). Think of each pile as the remaining value of a production machine and the operation corresponds to the passage of time and the depreciation of each machine.

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Austrian Solitaire
Austrian Solitaire as an application

As a good Austrian capitalist, you put aside money equivalent to the total depreciation of the system into a sinking fund for buying new machines (that cost $L$ each). So the next step of the Austrian Solitaire operation takes the collected dots plus the bank and buys as many machines as possible. That is, add as many parts $L$ as you can. Any remaining money is left in the bank.

Example with $n = 10$ and $L = 4$ from $(332|2)$.
Akin & Davis: “The economic interpretation shows that anything can become applied mathematics”

Let $P(n, L)$ denote the partitions of $n$ where each part is at most $L$. The generating function for $p(n, L) = |P(n, L)|$ is

$$\sum_{n=0}^{\infty} p(n, L)q^n = \prod_{i=1}^{L} \frac{1}{1 - q^i}$$

and direct formulas can be derived for small $L$.

A state $(\lambda|b)$ of Austrian solitaire satisfies $\lambda \in P(n - b, L)$. The defining operation for Austrian Solitaire is

$$A((\lambda|b)) = (L^{\lfloor \frac{t+b}{L} \rfloor}, \lambda_1 - 1, \ldots, \lambda_t - 1 \mid (b + t) \mod L)$$

where $\lfloor \cdot \rfloor$ denotes the least integer function.
Given $n$ and $L$, the Austrian Solitaire system consists of all states $(\lambda|0)$ for $\lambda \in P(n, L)$ and all iterated images under the operation $A$.

The $n = 6$, $L = 2$ system:

$$
\begin{align*}
1^6|0 & \rightarrow 222|0 & \rightarrow 2111|1 & \rightarrow 221|1 & \rightarrow 2211|0 \\
21111|0 & \\
\end{align*}
$$

Note that $(1^5|1)$ is not in the system...
The $n = 6$, $L = 5$ system:

\[
\begin{array}{cccccc}
33|0 & 2211|0 & 1^6|0 & 21^4|0 & \rightarrow 51|0 & \leftarrow 21^3 & \leftarrow 321|0 \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\
22|2 & \rightarrow 11|4 & \rightarrow 5|1 & \rightarrow 4|2 & & \rightarrow 2|4 & \leftarrow 3111|0 \\
\downarrow & & & & & & \\
222|0 & \rightarrow 111|3 & & \rightarrow 31|2 & \leftarrow 42|0 & \\
\end{array}
\]

$P(6, 5) = 10$ (all but (6)) then 9 with positive bank.
So Austrian Solitaire has three types of distinguished states: cycle, Garden of Eden, and positive bank states (needed to determine the size of a given system).
Austrian Solitaire types of states

So Austrian Solitaire has three types of distinguished states: cycle, Garden of Eden, and positive bank states (needed to determine the size of a given system).

But it seems two of these are connected:

The $n = 6$, $L = 2$ system has 2 GEs and 2 positive bank states.
The \( n = 6, L = 5 \) system has 9 of each.
Austrian Solitaire types of states

Connection GE states to positive bank states?

Path from each GE to a positive bank state? Uniqueness? Length?
A helpful statistic on $\lambda = (L^a, \lambda_{a+1}, \ldots, \lambda_t)$ where $\lambda_{a+1} < L$ is

$$h(\lambda) = \lambda_{a+1} + 1 + \cdots + \lambda_t + 1 = \lambda_{a+1} + \cdots + \lambda_t + t - a.$$ 

**Proposition 1: Valid positive bank states**

Suppose positive integers $n$, $L$, and $z$ with $1 \leq z \leq L - 1$ are given, and $\mu \in P(n - z, L)$. Then $(\mu|z)$ is a valid state for Austrian Solitaire if and only if $h(\mu) \leq n$.

For a positive bank state to be valid, it needs to have a pre-image under $A$ (the system starts from $P(n, L)$). The statistic $h(\mu)$ sums the parts required in the preimage state. E.g., the $(332|2)$ example with $n = 10$, $L = 4$ has $h(332) = 4 + 4 + 3 = 11$, not a valid state.
Write $\mu = (L^a, \mu a + 1, \ldots, \mu)$ where $\mu_{a+1} < L$. When $(\mu|z)$ satisfies $h(\mu) \leq n$, we can construct $\lambda \in P(n, L)$ such that $A((\lambda|0)) = (\mu|z)$:

$$\lambda = (\mu_{a+1} + 1, \ldots, \mu_t + 1, 1^{a(L+1)-t+z})$$

where $h(\mu) \leq n$ and the number of 1s is nonnegative.

This actually shows that every positive bank state has a bank 0 preimage. So only bank 0 states can be Garden of Eden states.
Proposition 2: At most one preimage per bank size

Suppose $n$ and $L$ are positive integers, $z$ satisfies $0 \leq z \leq L - 1$, and $\mu \in P(n - z, L)$ such that $(\mu|z)$ is a valid state of Austrian Solitaire. For each $y$ satisfying $0 \leq y \leq L - 1$, there is at most one valid state $(\lambda|y)$ with $A((\lambda|y)) = (\mu|z)$. Further, if $(\mu|z)$ has any preimage, then it has one with $y = 0$.

E.g., for $n = 6$, $L = 5$, $(2|4)$ has preimages $(3|3)$, $(31|2)$, and $(3111|0)$, nothing with bank 1 or 4.

Similar proof: Given $y$, the preimage bank size, explicitly construct $(\lambda|y)$, so it is unique if it exists.

For the statement about a bank 0 preimage, construct from $(\lambda|y)$ with $y \geq 1$ and $A((\lambda|y)) = (\mu|z)$ an explicit $\kappa \in P(n, L)$ with $A((\kappa|0)) = (\mu|z)$. 
In $n = 6$, $L = 5$, $(2|4)$ has preimages $(3|3)$, $(31|2)$, and $(3111|0)$.

In practice, two things can block constructing a $(\mu|z)$ preimage with a specified bank size $y$.

- The specified number of 1s in $\lambda$ could be negative: $(2|4)$ has no bank 4 preimage because

\[
\#1s = a(L + 1) - t + z - y = -1.
\]

- If $y \geq 1$, then $\lambda$ must also satisfy the condition given in Proposition 1: $(2|4)$ has no bank 1 preimage because the prescribed $\lambda = (3, 1, 1)$ has $h((3, 1, 1)) = 8 > 6$. 
Theorem 3: A bijection

Given positive integers $n$ and $L$, the number of Garden of Eden Austrian Solitaire states equals the number of positive bank Austrian Solitaire states.

Start with a Garden of Eden state and apply $A$ until you reach a positive bank state; associate those two. Chain can be arbitrarily long (e.g., there’s a length 3 connection in $n = 65$, $L = 5$).

If the path does not hit a positive bank state, then forms a bank 0 cycle (since $P(n, L) < \infty$) making some state have two bank 0 preimages, contradicting Proposition 2.

Essentially the propositions show $A$ is an invertible function between a GE state and its associated positive bank state.
Two questions reduced to one

Write \( ge(n, L) \) for the number of Garden of Eden states in the Austrian Solitaire system for parameters \( n \) and \( L \). We’ve shown that the size of the system is \( P(n, L) + ge(n, L) \).

**GE state characterization**

Given positive integers \( n \) and \( L \), the Garden of Eden states for Austrian Solitaire are the \((\lambda|0)\) with \( h(\lambda) > n \).

If positive bank, there’s a preimage. If \( h(\lambda) \leq n \), there’s a preimage.

The remaining task is to count these.
Counting GE states

\[ \lambda = (L^a, \lambda_{a+1}, \ldots, \lambda_t) \] and \( h(\lambda) > n \) imply \( t - a > aL \): there are more than \( aL \) parts of \( \lambda \) less than \( L \). Let \( \lambda_{aL+a+1} = s \); here is \( (\lambda_{a+1}, \ldots, \lambda_t) \).

\[ \begin{array}{c}
\mu \\
\downarrow \\
L-1-s \\
\times \\
s \\
\downarrow \\
aL \\
\end{array} \]

\[ \begin{array}{c} \mu \\
\downarrow \\
\lambda_{aL+a+1} \\
\end{array} \]

\( \mu \) fits in an \( (aL) \times (L - 1 - s) \) box; each part of \( \gamma \) is at most \( s \).
Counting GE states

The $q$-binomial coefficient $\binom{n+m}{m}$ counts partitions in an $n \times m$ box.

$$\sum_{n=0}^{\infty} ge(n, L) q^n = \sum_{a=0}^{\infty} q^{aL} \sum_{s=1}^{L-1} q^{s(aL+1)} \left[ \frac{aL + L - 1 - s}{aL} \right] \frac{1}{(q; q)_s}$$

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Austrian Solitaire
Counting GE states

\[
\sum_{n=0}^{\infty} ge(n, L)q^n = \sum_{a=0}^{\infty} q^{aL} \sum_{s=1}^{L-1} q^{s(aL+1)} \left[ L(a + 1) - s - 1 \right] \frac{1}{(q; q)_s}
= \cdots = \frac{1}{(q; q)_\infty} \sum_{s=1}^{L-1} \frac{(q^{L-s}; q)_\infty}{(q; q)_s} q^s \sum_{a=0}^{\infty} \frac{(q^{aL+1}; q)_\infty}{(q^{L(a+1)-s}; q)_\infty} q^{aL(s+1)}
= \cdots = \sum_{s=1}^{L-1} \sum_{m=0}^{L-1-s} \frac{(-1)^m q^{m+s} \binom{m}{2}}{(q; q)_s(q; q)_m(q; q)_{L-s-m-1}(1 - q^{L(m+s+1)})}
= \cdots = \sum_{k=1}^{L-1} \frac{(-1)^{k-1} q^{k+1}}{(q; q)_k(q; q)_{L-1-k}(1 - q^{L(k+1)})}.
\]

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Austrian Solitaire
For Austrian Solitaire:

- Characterize the cycle states. A few partial results in Kapil Bastola’s 2012 *undergraduate honors thesis* (guess the advisor), Proposition 2 here should be helpful. (Proposition 1 in his thesis, Theorem 3 was a conjecture.)

- Akin & Davis conjectured that an Austrian Solitaire system always has just one component (unlike Bulgarian Solitaire for $n = 8$), i.e., one just cycle. Intuition: Always adding parts $L$ means no reordering of parts, simpler setting. Observation: In a cycle, each state has a different size bank.

- Distances from GE states to positive bank states and to cycle states.
Open questions

For Bulgarian Solitaire:

- The system has multiple components whenever $n$ is 2 or more away from the nearest triangular number. Can we say anything about component sizes? Even the largest component? How can we identify which component includes a given partition?

- How far are GE partitions from cycles? Know minimum distance is 3. For $n = T_m$, the partition

$$
\gamma_m = (m-1, m-1, m-2, m-3, \ldots, 3, 2, 1, 1) \vdash n
$$

is at maximal distance $m(m-1)$ from the fixed point. Griggs & Ho 1998 conjectured maximum lengths for other $n$.

- How many partitions at the maximal distance from the cycle/fixed point? Whole spectrum by distance.
Open questions

For Bulgarian Solitaire:

- What is the interaction between conjugation of partitions and Bulgarian Solitaire? For $n = T_m$, Bentz 1987 showed that the path from $\gamma_m$ to the fixed point consists of nested conjugate pairs, e.g.,

2211 $\rightarrow$ 411 $\rightarrow$ 33 $\rightarrow$ $2^3$ $\rightarrow$ $31^3$ $\rightarrow$ 42 $\rightarrow$ 321

And similar questions for related operations; partition dynamics!