Non-commutative Gel'fand-Naĭmark Duality

Paolo Bertozzini

Department of Mathematics and Statistics - Thammasat University - Bangkok.

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dedicated to the memory of John E.Roberts and Renzo Cirelli

Abstract 1

We outline a new attempt to obtain a non-commutative generalization of the well-known Gel'fand-Naĭmark duality (between compact Hausdorff topological spaces and commutative unital C*-algebras), where "geometric spectra" for unital non-commutative C*-algebras can be described via "non-commutative spaceoids": suitable bundles of one-dimensional full C*-categories, equipped with a transition amplitude structure, satisfying certain saturation and uniformity conditions.

Abstract 2

This work is a joint collaboration with:

- Natee Pitiwan (Chulalongkorn University)
- Roberto Conti (Sapienza Università di Roma),

Outline

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- What are the "Spectra" of C*-algebras?

• Background

- Commutative Gel'fand-Naĭmark Duality:
  - Categories of Commutative (Unital) C*-algebras
  - Categories of Compact Hausdorff Spaces
  - Categorical Duality
- Previous Attempts:
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  - Convex Spaces of States as Duals
  - Quantales
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- Review of Commutative Gel'fand-Naĭmark Duality

C*-algebras 1

Defined by I.Gel'fand-M.Naĭmark in 1943, are a "rigid" blend of algebra and topology, basic in functional analysis, non-commutative geometry and quantum physics.

A **complex unital C*-algebra** $(\mathcal{C}, \circ, *, +, \cdot, \| \ \|)$ is given by:

- a complex associative unital involutive algebra:
  - a vector space $(\mathcal{C}, +, \cdot)$ over $\mathbb{C}$,
  - an associative unital bilinear multiplication $\circ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
  - a conjugate-linear antimutiplicative involution $* : \mathcal{C} \to \mathcal{C}$,
- a norm $\| \ \| : \mathcal{C} \to \mathbb{R}$ such that:
  - completeness: $(\mathcal{C}, +, \cdot, \| \ \|)$ is a Banach space,
  - submultiplicativity: $\|x \circ y\| \leq \|x\| \cdot \|y\|$, for all $x, y \in \mathcal{C}$,
  - C*-property: $\|x^* \circ x\| = \|x\|^2$, for all $x \in \mathcal{C}$.

A C*-algebra is Abelian (or **commutative**) if $x \circ y = y \circ x$.

C*-algebras 2

- A **unital $*$-homomorphism** is a map $\phi : \mathcal{A} \to \mathcal{B}$ between unital C*-algebras $\mathcal{A}, \mathcal{B}$ such that for all $x, y \in \mathcal{A}$:

$$\phi(x \circ_{\mathcal{A}} y) = \phi(x) \circ_{\mathcal{B}} \phi(y), \quad \text{and} \quad \phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}.$$

- Examples of unital C*-algebras:
  - $\mathcal{L}(H)$ set of linear continous maps $H \xrightarrow{T} H$ in a complex Hilbert space H with composition, adjunction and operator norm $\|T\| := \inf\{k \in \mathbb{R}_+ \mid \|T(h)\|_H \leq k\|h\|_H, \forall h \in H\}$. Every unital C*-algebra is an operator-norm-closed unital $*$-subalgebra of $\mathcal{L}(H)$ for a certain H.
  - Every Abelian unital C*-algebra is of the form $C(X; \mathbb{C})$: the set of complex-valued continuous functions $X \xrightarrow{f} \mathbb{C}$ over a compact Hausdorff topological space X with pointwise multiplication and conjugation and norm $\|f\| := \sup_{p \in X} |f(p)|$.

Categories

A **category** $\mathscr{C}$ consists of:

- a **quiver**: a pair of **source**/**target** maps $\mathscr{C}^0 \xleftarrow{s} \mathscr{C}^1 \xrightarrow{t} \mathscr{C}^0$ from a class $\mathscr{C}^1$ of **morphisms** to a class $\mathscr{C}^0$ of **objects**,
- an **identity** map $\mathscr{C}^0 \xrightarrow{\iota} \mathscr{C}^1$ that to every object $A \in \mathscr{C}^0$ associates its identity morphisms $\iota_A \in \mathscr{C}^1$ such that $s(\iota_A) = A = t(\iota_A)$,
- a partially defined **composition** map that to every pair of morphisms $f, g \in \mathscr{C}^1$ such that $t(g) = s(f)$ associates a new morphism $f \circ g \in \mathscr{C}^1$ with $s(f \circ g) = s(g)$, $t(f \circ g) = t(f)$,

that further satisfies the following algebraic axioms:

- **associativity**: $(f \circ g) \circ h = f \circ (g \circ h)$, whenever (one of) the two terms are defined,
- **unitality**: $f \circ \iota_A = f = \iota_B \circ f$, whenever $f \in \mathrm{Hom}_{\mathscr{C}}(A, B) := \{x \in \mathcal{C}^1 \mid s(x) = A, t(x) = B\}$.

Functors

A **covariant functor** $\mathscr{S} \xrightarrow{\Gamma} \mathscr{A}$ between the categories $\mathscr{S}$ and $\mathscr{A}$ is a pair of maps $\Gamma^1 : \mathscr{S}^1 \to \mathscr{A}^1$, $\Gamma^0 : \mathscr{S}^0 \to \mathscr{A}^0$ making commutative the following diagrams:

$$\begin{array}{ccc} \mathscr{S}^1 & \xrightarrow{\Gamma^1} & \mathscr{A}^1 \\ {\scriptstyle s_{\mathscr{S}}}\downarrow & & \downarrow{\scriptstyle s_{\mathscr{A}}} \\ \mathscr{S}^0 & \xrightarrow{\Gamma^0} & \mathscr{A}^0 \end{array} \qquad \begin{array}{ccc} \mathscr{S}^1 & \xrightarrow{\Gamma^1} & \mathscr{A}^1 \\ {\scriptstyle t_{\mathscr{S}}}\downarrow & & \downarrow{\scriptstyle t_{\mathscr{A}}} \\ \mathscr{S}^0 & \xrightarrow{\Gamma^0} & \mathscr{A}^0 \end{array}$$

and that also satisfy the following unital homomorphim axioms:

$$\Gamma^1(f \circ_{\mathscr{S}} g) = \Gamma^1(f) \circ_{\mathscr{A}} \Gamma^1(g), \qquad \Gamma^1(\iota^{\mathscr{S}}_A) = \iota^{\mathscr{A}}_{\Gamma^0(A)}.$$

A **contravariant functor** will intertwine sources with targets in the diagrams and will satisfy axioms for unital **anti**-homomorphism.

Natural Transformations

A **natural transformation** $\Gamma \xrightarrow{\mathfrak{G}} \Xi$ between two covariant (or contravariant) functors $\mathscr{S} \xrightarrow{\Gamma,\ \Xi} \mathscr{A}$ consists of a map $\mathfrak{F} : \mathscr{S}^0 \to \mathscr{A}^1$ such that for every morphism $A \xrightarrow{f} B$ in $\mathscr{S}^1$ the following diagram in $\mathscr{A}$ commutes:[1]

$$\begin{array}{ccc} \Gamma^0(A) & \xrightarrow{\mathfrak{F}_A} & \Xi^0(A) \\ {\scriptstyle \Gamma^1(f)}\downarrow & & \downarrow{\scriptstyle \Xi^1(f)} \\ \Gamma^0(B) & \xrightarrow[\mathfrak{F}_B]{} & \Xi^0(B) \end{array}$$

A **natural isomorphism** is a natural transformation such that $\mathfrak{F}_A$ is **invertible** in $\mathscr{A}^1$, for all $A \in \mathscr{S}^0$.

[1] In the contravariant case the direction of vertical arrows is reversed.

Duality

An **(anti-)isomorphism** of categories is given by a pair of

co(ntra)variant functors $\mathscr{S} \underset{\Sigma}{\overset{\Gamma}{\rightleftarrows}} \mathscr{A}$ such that $\Sigma \circ \Gamma = I_{\mathscr{S}}$ and

$\Gamma \circ \Sigma = I_{\mathscr{A}}$, where $I_{\mathscr{C}}$ denotes the identity functor of $\mathscr{C}$.

An **equivalence** between categories is a pair of covariant functors

$\mathscr{S} \underset{\Sigma}{\overset{\Gamma}{\rightleftarrows}} \mathscr{A}$ with two natural isomorphisms

$$I_{\mathscr{A}} \xrightarrow{\mathfrak{G}} \Gamma \circ \Sigma, \quad I_{\mathscr{S}} \xrightarrow{\mathfrak{E}} \Sigma \circ \Gamma.$$

When the above functors Γ, Σ are contravariant, we say that we have a **duality** between the categories $\mathscr{S}$ and $\mathscr{A}$.

Gel'fand-Naĭmark Duality 1

Theorem ([2])

There is a duality between the categories:

- $\mathscr{A}_o$ *of unital ∗-homomorphisms between commutative unital C*-algebras,*
- $\mathscr{S}_o$ *of continuous maps betwen compact Hausdorff topological spaces.*

[2] Gel'fand I (1941) Normierte Ringe *Mat Sbornik N S* 51(9):3-24
Gel'fand I, Naĭmark M (1943) On the Embedding of Normed Rings into the Ring of Operators in Hilbert Space *Math Sbornik* 12:197-213

Gel'fand-Naĭmark Duality 2

- The functor $\Gamma_o : \mathscr{S}_o \to \mathscr{A}_o$ associates to every compact Hausdorff space X the commutative unital C*-algebra $\Gamma_o(X)$ of continuous complex-valued functions on X (with pointwise multipication and conjugation and maximum modulus norm).
- The functor $\Sigma_o : \mathscr{A}_o \to \mathscr{S}_o$ associates to every commutative unital C*-algebra $\mathcal{A}$ its **Gel'fand spectrum** $\Sigma_o(\mathcal{A}) := \{\omega : \mathcal{A} \to \mathbb{C} \mid \omega \text{ is a unital } *\text{-homomorphism}\}$ equipped with the (compact Hausdorff) weak*-topology: the weakest topology making continuous for all $x \in \mathcal{A}$ the **Gel'fand transforms** $\hat{x} : \Sigma_o(\mathcal{A}) \to \mathbb{C}$, $\hat{x}(\omega) := \omega(x)$.

Gel'fand-Naĭmark Duality 3

- The **Gel'fand transform** $I_{\mathscr{A}_o} \xrightarrow{\mathfrak{G}^o} \Gamma_o \circ \Sigma_o$ is the natural isomorphism that, for every $\mathcal{A} \in \mathscr{A}_o^0$, associates the unital $*$-isomorphism of C*-algebras $\mathcal{A} \xrightarrow{\mathfrak{G}^o_{\mathcal{A}}} \Gamma_o \circ \Sigma_o(\mathcal{A})$ given by: $\mathfrak{G}^o_{\mathcal{A}} : x \mapsto \hat{x}$, for $x \in \mathcal{A}$,
- The **evaluation transform** $I_{\mathscr{S}_o} \xrightarrow{\mathfrak{E}^o} \Sigma_o \circ \Gamma_o$ is the natural isomorphism that, for every $X \in \mathscr{S}_o^0$, associates the homeomorphism $X \xrightarrow{\mathfrak{E}^o_X} \Sigma_o \circ \Gamma_o(X)$ given by: $\mathfrak{E}^o_X : p \mapsto \mathrm{ev}_p$, for $p \in X$, where $\mathrm{ev}_p : \Gamma_o(X) \to \mathbb{C}$ is the p-evaluation map $\mathrm{ev}_p : \sigma \mapsto \sigma(p)$, for all $\sigma \in \Gamma_o(X)$.

Gel'fand-Naĭmark Duality 4

This topological version of Descartes's algebraization of geometry is the usual starting point of **non-commutative geometry**:

- since a commutative unital C*-algebra "is" a compact Hausdorff topological space, we will think of non-commutative C*-algebras as (duals of) "quantum topological spaces",
- we can work in the "dual" category of unital C*-algebras, as a substitute for a missing category of "quantum compact Hausdorff topological spaces".

Without the intention of undermining the basic usefulness of such "dual" point of view, it is the purpose of the present research work to provide a "geometrical/spectral" counterpart to non-commutative unital C*-algebras.

C*-categories

Horizontal categorification of C*-algebras defined by J.Roberts.

A **C*-category** $(\mathcal{C}, \circ, *, +, \cdot, \| \, \|)$ is given by:

- an involutive algebroid $(\mathcal{C}, \circ, *, +, \cdot)$ over $\mathbb{C}$:
  - a category $(\mathcal{C}, \circ)$, with identities $\mathcal{C}^0 \subset \mathcal{C}$,
  - a contravariant functor $* : \mathcal{C} \to \mathcal{C}$ acting trivially on $\mathcal{C}^0$,
  - $\forall A, B \in \mathcal{C}^0$, $(\mathcal{C}_{AB}, +, \cdot)$, $\mathcal{C}_{AB} := \mathrm{Hom}_{\mathcal{C}}(B, A)$, are complex vector spaces on which $\circ$ is bilinear and $*$ is conjugate-linear,
- equipped with a norm $\| \, \| : \mathcal{C} \to \mathbb{R}$ such that:
  - completeness: $(\mathcal{C}_{AB}, +, \cdot)$ are Banach spaces, $\forall A, B \in \mathcal{C}^0$,
  - submultiplicativity: $\|x \circ y\| \leq \|x\| \cdot \|y\|$,
  - C*-property: $\|x^* \circ x\| = \|x\|^2$, for all $x \in \mathcal{C}$,
  - positivity: for all $x \in \mathcal{C}$, the element $x^* \circ x$ is positive in the unital C*-algebra $\mathcal{C}_{s(x)s(x)}$, where $s(x) \xrightarrow{x} t(x)$.

A C*-category is **full** if all the bimodules ${}_{\mathcal{C}_{AA}}(\mathcal{C}_{AB})_{\mathcal{C}_{BB}}$ are full.

Fell Bundles

A "bundle version" of C*-categories developed by J.Fell.

A **Fell bundle** is a Banach bundle ref $(\mathcal{E}, \pi, \mathcal{X})$ such that:

- $(\mathcal{E}, \circ, *)$ and $(\mathcal{X}, \circ, *)$ are topological involutive categories,
- $\pi : \mathcal{E} \to \mathcal{X}$ is a $*$-functor,
- restricted to the fibers $\mathcal{E}_p := \pi^{-1}(p)$, for $p \in \mathcal{X}$, $\circ$ is bilinear and $*$ is conjugate-linear,
- $\|x \circ y\| \leq \|x\| \cdot \|y\|$, for all composable $x, y \in \mathcal{E}$,
- $\|x^* \circ x\| = \|x\|^2$, for all $x \in \mathcal{E}$ and
- $x^* \circ x$ is positive whenever it belongs to a C*-algebraic fiber.[3]

A Fell bundle is **saturated** if $\mathcal{E}_p \circ \mathcal{E}_q$ is dense in $\mathcal{E}_{p \circ q}$.

[3] If $(\mathcal{X}, \circ, *)$ is inverse involutive category ($p \circ p^* \circ p = p \in \mathcal{X}$) or a groupoid, simply require $x^* \circ x$ positive in the C*-algebra $\mathcal{E}_{\pi(x^* \circ x)}$.

Banach Bundles

A **Banach bundle** is a bundle $(\mathcal{E}, \pi, \mathcal{X})$, i.e. a continous open surjective map $\pi : \mathcal{E} \to \mathcal{X}$, whose total space is equipped with:

- a partially defined continuous binary operation of addition $+ : \mathcal{E} \times_{\mathcal{X}} \mathcal{E} \to \mathcal{E}$, with domain the subset $\mathcal{E} \times_{\mathcal{X}} \mathcal{E} := \{(x, y) \in \mathcal{E} \times \mathcal{E} \mid \pi(x) = \pi(y)\}$,
- a continuous operation of multiplication by scalars $\cdot : \mathbb{K} \times \mathcal{E} \to \mathcal{E}$,
- a continuous "norm" $\| \cdot \| : \mathcal{E} \to \mathbb{R}$, such that:
  - for all $x \in \mathcal{X}$, the fiber $\mathcal{E}_x := \pi^{-1}(x)$ is a complex Banach space $(\mathcal{E}_x, +, \cdot)$ with the norm $\| \cdot \|$,
  - for all $x_o \in \mathcal{X}$, the family $U^{\mathcal{O},\epsilon}_{x_o} = \{e \in \mathcal{E} \mid \|e\| < \epsilon,\ \pi(e) \in \mathcal{O}\}$, where $\mathcal{O} \subset \mathcal{X}$ is an open set containing $x_o \in \mathcal{X}$ and $\epsilon > 0$, is a fundamental system of neighbourhoods of $0 \in E_{x_o}$.

back

• Previous/Other Attempts

Several lines of approach to a non-commutative extension of Gel'fand-Naĭmark duality have been attempted.

We give here a brief guide to the complex literature on this topic.

Bundle Duals / Sectional Representations

J.M.G.Fell probably initiated the trend of sectional reconstruction of a C*-algebra $\mathcal{A}$ using, as dual, a suitable bundle over its spectral space $\hat{\mathcal{A}}$ (the set of equivalence classes of unitarily equivalent $*$-representations).[4]
Results in similar directions were produced by **J.Tomiyama**.[5]
J.M.G.Fell also pioneered the definition of the now called Fell-bundles.[6]

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[4] Fell JMG (1961) The Structure of Algebras of Operator Fields *Acta Math* 106:233-280

[5] Tomiyama J (1962) Topological Representations of C*-algebras *Tohoku Math J* 14(2):187-204

[6] Fell J, Doran R (1998) *Representations of C*-algebras, Locally Compact Groups and Banach $*$-algebraic Bundles* Vol 1-2 Academic Press

Bundle Duals / Sectional Representations

In the celebrated (but vastly ignored) **J.Dauns-K.H.Hofmann** theorem[7] the algebra is reconstructed via continous sections of a bundle simple C*-algebras (see also **J.Migda**[8]).

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[7] Dauns J, Hofmann K-H (1968) Representations of Rings by Sections *Mem Amer Math Soc* 83 AMS

Hofmann K H (1972) Representation of Algebras by Continuous Sections *Bull Amer Math Soc* 78(3):291-373

Hofmann K H (1972) Some Bibliograpical Remarks on: "Representation of Algebras by Continuous Sections" *Recent Advances in the Representation Theory of Rings and C*-Algebras* 177-182 (eds) Hofmann K H, Liukkonen J R *Memoirs Amer Math Soc* 148 (1974)

Hofmann K H (2011) The Dauns-Hofmann Theorem Revisited *Journal of Algebra and Its Applications* 10(1):29-37

[8] Midga J (1993) Non-commutative Gelfand-Naimark Theorem *Comment Math Univ Carolin* 34(2):253-255

Structured Pure State Space Duals / Functional

Functional representations of C*-algebras via continuous functions on generalized spectra consisting of (pure) states equipped with extra structures (transition probability, Poisson, ...) started with **R.Kadison**[9] and were subsquently considered by **F.Schultz**,[10] **P.Kruszyński-S.Woronowicz**,[11] (see also **I.Fujimoto**).[12]
A duality was essentially obtained by **N.Landsman**.[13]

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[9] Kadison R-V (1951) A Representation Theory for Commutative Topological Algebra *Memoires Amer Math Soc* 7

[10] Schultz F (1982) Pure States as a Dual Object for C*-algebras *Commun Math Phys* 82:497-509

[11] Kruszyński P, Woronowicz S (1982) A Noncommutative Gelfand Naimark Theorem *J Operator Theory* 8:361-389

[12] Fujimoto I (1998) A Gelfand-Naimark Theorem for C*-algebras *Pacific J Math* 184(1):95-119

[13] Landsman N (1997) Poisson Spaces with a Transition Probability *Reviews in Mathematical Physics* 9(1):29-57

Bundle of Pure States Duals / Functional

In the mostly ignored work by **R.Cirelli-A.Maniá-L.Pizzocchero**[14] spectra are projective Kähler bundles over the spectrum $\hat{\mathcal{A}}$.

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[14] Cirelli R, Lanzavecchia P, Manià A (1983) Normal Pure States of the von Neumann Algebra of Bounded Operators as Kähler Manifold *J Phys A: Math Gen* 16:3829-3835

Cirelli R, Lanzavecchia P (1984) Hamiltonian Vector Fields in Quantum Mechanics *Il Nuovo Cimento* 79(2):271-283

Abbati M-C, Cirelli R, Lanzavecchia P, Maniá A (1984) *Il Nuovo Cimento B* 83(1):43-60

Cirelli R, Manià A, Pizzocchero L (1990) Quantum Mechanics as an Infinite-dimensional Hamiltonian System with Uncertainty Structure: Part I *J Math Phys* 31:2891-2897

Cirelli R, Manià A, Pizzocchero L (1990) Quantum Mechanics as an Infinite-dimensional Hamiltonian System with Uncertainty Structure: Part II *J Math Phys* 31:2898-2903

Cirelli R, Manià A, Pizzocchero L (1994) A Functional Representation for Non-commutative C*-algebras *Rev Math Phys* 6(5):675-697

NC-topology on Maximal Ideals / Functional

C.Akemann[15] was probably the first to describe the dual of a C*-algebra via maximal left-ideals with a "non-commutative" form of topology.

Reformulations of commutative Gel'fand-Naĭmark duality via locales by **B.Banaschewski-C.Mulvey**[16] (for constructive versions see also T.Coquand-B.Spitters[17]) further inpired the usage of quantales as duals of non-commutative C*-algebras.[18]

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[15] Akemann C (1971) A Gelfand Representation Theory for C*-algebras *Pacific Journal of Mathematics* 39 (1):1-11

[16] Banaschewski B, Mulvey CJ (2000) The Spectral Theory of Commutative C*-algebras *Quaestiones Mathematicae* 23:425-464

[17] Coquand T, Spitters B (2009) Constructive Gelfand Duality for C*-algebras *Mathematical Proceedings of the Cambridge Philosophical Society* 147:339-344

[18] Banaschewski B, Mulvey CJ (2006) A Globalisation of the Gelfand Duality Theorem *Annals of Pure and Applied Logic* 137:62-103

Quantales as Duals

The study of "point free" non-commutative topologies via quantales, as duals of C*-algebras, has been further pursued by **F.Borceux-J.Rosický-G.Van den Bossche**[19] **C.J.Mulvey-J.W.Pelletier**[20] and **D.Kruml-J.W.Pelletier-P.Resende-J.Rosický**.[21]

[19] Borceux F, Rosický J, Van den Bossche G (1989) Quantales and C*-algebras *J London Math Soc* 40:398-404

[20] Mulvey CJ, Pelletier JW (2002) On the Quantisation of Spaces *J Pure Appl Algebra* 175:289-325

[21] Kruml D, Pelletier J W, Resende P, Rosický J (2003) On Quantales and Spectra of C*-algebras *Appl Categ Structures* 11:543-560

Kruml D, Resende P (2004) On Quantales that Classify C*-algebras *Cah Topol Geom Differ Categ* 45:287-296

Resende P (2007) Étale Groupoids and their Quantales *Advances in Mathematics* 208(1):147-209

Topos Theory Approaches

Several works suggest to reconstruct C*-algebras via the Grothendieck topoi of their commutative subalgebras

- **A.Döring**[22] (based on works by C.Isham-J.Butterfield).
- **C.Heunen-N.Landsman-B.Spitters-S.Wolters**[23]

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[22] Döring A (2012) Generalised Gelfand Spectra of Nonabelian Unital C*-Algebras arXiv:1212.2613
Döring A (2012) Flows on Generalised Gelfand Spectra of Nonabelian Unital C*-Algebras and Time Evolution of Quantum Systems arXiv:1212.4882

[23] Heunen C, Landsman N, Spitters B (2009) A Topos for Algebraic Quantum Theory *Communications in Mathematical Physics* 291:63-110
Heunen C, Landsman NP, Spitters B, Wolters S (2011) The Gelfand Spectrum of a Noncommutative C*-algebra: a Topos-theoretic Approach *J Austr Math Soc* 90:39-52
Wolters S (2013) A Comparison of Two Topos-theoretic Approaches to Quantum Theory *Communications in Mathematical Physics* 317(1):3-53

Other Categorical Sheaves / Topos Theory Approaches

Among the ongoing recent efforts towards Gel'fand-Naĭmark duality using topoi or sheaves we mention:

- **S.Henry**[24]
- **C.Flori-T.Fritz**[25]

[24] Henry S (2014) Localic Metric Spaces and the Localic Gelfand Duality `arXiv:1411.0898 [math.CT]`
Henry S (2014) Constructive Gelfand Duality for Non-unital Commutative C*-algebras `arXiv:1412.2009 [math.CT]`
Henry S (2015) Toward a Non-commutative Gelfand Duality: Boolean Locally Separated Toposes and Monoidal Monotone Complete C*-categories `arXiv:1501.07045 [math.CT]`

[25] Flori C, Fritz T (2015) (Almost) C*-algebras as Sheaves with Self Action arXiv:1512.01669

Other Approaches

Apart from the extremely vast literature on classification of C*-algebras, some quite recent attempts to produce Gel'fand-Naĭmark dualities, at least for some reasonable classes of C*-algebras have been put forward by:

- **C.Heunen-M.Reyes**[26]
- **N.de Silva**[27]

[26] Heunen C, Reyes ML (2014) Active Lattices Determine AW*-algebras *Journal of Mathematical Analysis and Applications* 416:289-313

Heunen C, Reyes ML On Discretization of C*-algebras arXiv:1412.1721

[27] de Silva N (2014) From Topology to Noncommutative Geomtery: K-theory arXiv:1408.1170

The Present Approach

In our proposed approach, C*-algebras are reconstructed by sections of a bundle in the same tradition of J.M.G.Fell, J.Dauns-K.H.Hofmann, R.Cirelli-A.Maniá-L.Pizzocchero and N.Landsman.

Contrary to these previous cases,

- our bundles have only one-dimensional fibers,
- all differential geometric features (Kähler / Poisson structures) are eliminated from the spectrum and "substituted" by a horizontal categorification of the base of the bundle.

Direct inspirational input of this project comes from:

- W.Heisenberg / A.Connes,
- R.Feynmann / L.Crane.

• Basic Constructions

- The Transition Amplitude Bundle
- The Transition Amplitude Space
- Non-commutative Spaceoids

- The Transition Amplitude Bundle

The Bundle of Pure States

- The family $\mathcal{S}_\mathcal{A}$ of states of unital C*-algebra $\mathcal{A}$ consists of all the linear maps $\omega : \mathcal{A} \to \mathbb{C}$ that are
  - positive: $\omega(x^* \circ x) \in \mathbb{C}_+$, for all $x \in \mathcal{A}$,
  - normalized: $\omega(1_\mathcal{A}) = 1_\mathbb{C}$.
- By Gel'fand-Naĭmark-Segal theorem, every state ω induces a representation $\pi_\omega : \mathcal{A} \to \mathcal{L}(H_\omega)$ and a unit vector $\xi_\omega \in H_\omega$ such that $\omega(x) = \langle \xi_\omega \mid \pi_\omega(x)\xi_\omega \rangle_{H_\omega}$, for all $x \in \mathcal{A}$.
- Let $\mathcal{P}_\mathcal{A} \subset \mathcal{S}_\mathcal{A}$ denote the family of pure states of $\mathcal{A}$: these are those states $\omega \in \mathcal{S}_\mathcal{A}$ such that π_ω is irreducible.
- $\mathcal{P}_\mathcal{A} \subset \mathcal{S}_\mathcal{A} \subset \mathcal{A}^*$ is equipped with the weak*-topology: the weakest topology making continuous all the maps $\hat{x} : \omega \mapsto \omega(x)$, for all $x \in \mathcal{A}$.
- $\mathcal{P}_\mathcal{A}$ is a bundle over $\hat{\mathcal{A}}$, the usual "spectrum of $\mathcal{A}$", that is the quotient space of $\mathcal{P}_\mathcal{A}$ under the equivalence relation of unitary equivalence of irreducible GNS-representations.

The Base Spectrum

The base space of our spectra is just a fiberwise **horizontal categorification** of the previous bundle $\mathcal{P}_{\mathcal{A}}$ over $\hat{\mathcal{A}}$:
instead of considering only the points $\omega \in \mathcal{P}_{\mathcal{A}}$, we consider all possible 1-arrows (ordered pairs) between pure states in the same fiber of bundle $\mathcal{P}_{\mathcal{A}}$.

The fiberwise product

$$\mathcal{P}_{\mathcal{A}} \times_{\hat{\mathcal{A}}} \mathcal{P}_{\mathcal{A}} := \{(\omega, \rho) \mid [\pi_\omega] = [\pi_\rho]\} \subset \mathcal{P}_{\mathcal{A}} \times \mathcal{P}_{\mathcal{A}},$$

with the topology induced by the product of the weak*-topology on $\mathcal{P}_{\mathcal{A}}$, is a bundle over $\hat{\mathcal{A}}$ with pair groupoids as fibers.

$\mathcal{P}_{\mathcal{A}} \times_{\hat{\mathcal{A}}} \mathcal{P}_{\mathcal{A}}$ is a **bundle gerbe** of pair groupoids over $\hat{\mathcal{A}}$ and is the **base spectrum of** $\mathcal{A}$.

The Von Neumann Enveloping Bundle

- If $\omega \in \mathcal{P}_{\mathcal{A}}$, the GNS-representation π_ω is irreducible, hence $\pi_\omega(\mathcal{A})'' = \mathcal{L}(H_\omega)$. Furthermore, if $[\omega] = [\rho]$, $\exists U : H_\omega \to H_\rho$ unitary such that $\pi_\rho(x) = U\pi_\omega(x)U^*$, for all $x \in \mathcal{A}$.
- U is unique up to a phase, since $U_1{}^* U_2 \in \pi_\omega(\mathcal{A})' = \mathbb{C} \cdot 1_{H_\omega}$, but the unital $*$-isomorphim $\pi_\omega(\mathcal{A})'' \xrightarrow{\mathrm{Ad}_U} \pi_\rho(\mathcal{A})''$ is unique.
- For every $p \in \hat{\mathcal{A}}$ consider the pair groupoid with objects $\mathcal{L}(H_\omega)$, with $[\omega] = p$, and 1-arrows the unique unital $*$-isomorphisms Ad_U induced by the unitaries U intertwining the given representations and construct the W*-algebra $\mathcal{A}''_p$ of orbits of such groupoid ($\mathcal{A}''_p \simeq \mathcal{L}(H_\omega), \forall \omega \in p$).
- The bundle of (type I factors) W*-algebras $\mathcal{A}''_p$ over $\hat{\mathcal{A}}$ is the **Von Neumann enveloping bundle** of $\mathcal{A}$. $\bigoplus_{p \in \hat{\mathcal{A}}} \mathcal{A}''_p$ coincides with the atomic part of the Arens W*-envelope $\mathcal{A}^{**}$ of $\mathcal{A}$.

The Total Spectrum

- There is a natural embedding of $(\mathcal{P}_\mathcal{A})_p$ into $\mathcal{A}''_p$ that to every $\omega \in p$ associates $|\,\omega\,\rangle\langle\,\omega\,|$ the (orbit of the) one-dimensional projector $|\,\xi_\omega\,\rangle\langle\,\xi_\omega\,|$ in $\mathcal{L}(H_\omega) \simeq \mathcal{A}''_p$.
- For every $\omega, \rho \in (\mathcal{P})_p$, the "corner space" $|\,\omega\,\rangle\langle\,\omega\,|\,\mathcal{A}''_p\,|\,\rho\,\rangle\langle\,\rho\,|$ is one-dimensional.
- The **total spectrum** $\mathcal{E}_\mathcal{A}$ **of** $\mathcal{A}$ is the disjoint union $\biguplus_{(\omega,\rho)\in\mathcal{P}_\mathcal{A}\times_{\hat{\mathcal{A}}}\mathcal{P}_\mathcal{A}} |\,\omega\,\rangle\langle\,\omega\,|\,\mathcal{A}''_p\,|\,\rho\,\rangle\langle\,\rho\,|$ as a bundle over the base spectrum $\mathcal{P}_\mathcal{A}\times_{\hat{\mathcal{A}}}\mathcal{P}_\mathcal{A}$.
- $\mathcal{E}_\mathcal{A}$ is an involutive category: in each fiber $(\mathcal{E}_\mathcal{A})_p$ the composition and the involution are, for $x, y \in \mathcal{A}''_p$, $\omega, \rho, \eta \in p$:

$$
\begin{aligned}
(|\,\omega\,\rangle\langle\,\omega\,|\,x\,|\,\rho\,\rangle\langle\,\rho\,|) \circ (|\,\rho\,\rangle\langle\,\rho\,|\,y\,|\,\eta\,\rangle\langle\,\eta\,|) \\
:= |\,\omega\,\rangle\langle\,\omega\,|\,x\,|\,\rho\,\rangle\langle\,\rho\,|\,y\,|\,\eta\,\rangle\langle\,\eta\,|, \\
(|\,\omega\,\rangle\langle\,\omega\,|\,x\,|\,\rho\,\rangle\langle\,\rho\,|)^* := |\,\rho\,\rangle\langle\,\rho\,|\,x^*\,|\,\omega\,\rangle\langle\,\omega\,|\,.
\end{aligned}
$$

The Non-commutative Gel'fand Transform

- Every $x \in \mathcal{A}$ determines a section of the Von Neumann enveloping bundle $x \mapsto \pi_p(x)$ (the orbit of $\pi_\omega(x)$).
- Every $x \in \mathcal{A}$ induces a section $\hat{x}$ of the total spectrum bundle $\mathcal{E}_\mathcal{A}$ over $\mathcal{P}_\mathcal{A} \times_{\hat{\mathcal{A}}} \mathcal{P}_\mathcal{A}$ given, for all $p \in \hat{\mathcal{A}}$ and for all $\omega, \rho \in p$ by:

$$\hat{x}(\omega, \rho) := \mid \omega \rangle \langle \omega \mid \pi_p(x) \mid \rho \rangle \langle \rho \mid .$$

The section $\hat{x}$ is the **Gel'fand transform of** $x \in \mathcal{A}$.

The Transition Amplitude Bundle

- The total spectrum $\mathcal{E}_{\mathcal{A}}$ of $\mathcal{A}$ becomes a **Fell bundle** with the topology induced by the bases of "tubular neighborhoods" $\mathcal{U}_{\mathcal{O}}^{x,\epsilon}$ of any point $e_o \in (\mathcal{E}_{\mathcal{A}})_{(\omega_o,\rho_o)}$ defined as follows: for every open neighborhood $\mathcal{O} \subset \mathcal{P}_{\mathcal{A}} \times_{\hat{\mathcal{A}}} \mathcal{P}_{\mathcal{A}}$ of (ω_o, ρ_o), for every $x \in \mathcal{A}$ such that $\hat{x}(\omega_o, \rho_o) = e_o$, for every $\epsilon > 0$,

$$\mathcal{U}_{\mathcal{O}}^{x,\epsilon} := \{e \in \mathcal{E}_{\mathcal{A}} \mid \forall(\omega,\rho) \in \mathcal{O},\ \|\hat{x}(\omega,\rho) - e\| < \epsilon\}.$$

  The Fell bundle $\mathcal{E}_{\mathcal{A}}$ is the **transition amplitude bundle of** $\mathcal{A}$.
- For every $p \in \hat{\mathcal{A}}$, the sub-bundle $(\mathcal{E}_{\mathcal{A}})_p$ is actually a one-dimensional C*-category with objects $(\mathcal{P}_{\mathcal{A}})_p$, hence the total spectrum of $\mathcal{A}$ can be described alternatively as a bundle of one-dimensional C*-categories.

Bundle Uniformity on the Transition Amplitude Bundle

Since the spaces $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{B}_{\mathcal{A}} := \mathcal{P}_{\mathcal{A}} \times_{\hat{\mathcal{A}}} \mathcal{P}_{\mathcal{A}}$ are not usually compact, we will need to consider uniformly continuous sections.

- There is a standard uniform structure $\mathscr{U}_{\mathcal{B}_{\mathcal{A}}}$ on the space $\mathcal{B}_{\mathcal{A}} := \mathcal{P}_{\mathcal{A}} \times_{\hat{\mathcal{A}}} \mathcal{P}_{\mathcal{A}} \subset \mathcal{P}_{\mathcal{A}} \times \mathcal{P}_{\mathcal{A}} \subset \mathcal{A}^* \times \mathcal{A}^*$ obtained by restricting the product of the uniform structure induced by the weak*-topology on $\mathcal{A}^*$.
- A **bundle uniformity** $\mathscr{U}_{\mathcal{E}_{\mathcal{A}}}$ for the transition amplitude bundle is given by the filter generated on $\mathcal{E}_{\mathcal{A}} \times \mathcal{E}_{\mathcal{A}}$ by this filterbase of subsets of $\mathcal{E}_{\mathcal{A}} \times_{\mathcal{B}_{\mathcal{A}}} \mathcal{E}_{\mathcal{A}}$:

$$U^{\epsilon} := \{(e_1, e_2) \in \mathcal{E}_{\mathcal{A}} \times_{\mathcal{B}_{\mathcal{A}}} \mathcal{E}_{\mathcal{A}} \mid \|e_1 - e_2\|_{\mathcal{E}_{\mathcal{A}}} < \epsilon\}, \quad \epsilon > 0.$$

  Note that this uniformity does **not** induce the already defined topology on the total space $\mathcal{E}$ of the transition amplitude bundle, hence the total space is not itself a uniform space!

- The Transition Amplitude Space

The Gauge Section of a Unital C*-algebra

Every unital C*-algebra determines a very special **gauge section** of its transition amplitudes bundle $\mathcal{E}_{\mathcal{A}}$: the Gel'fand transform $\hat{1}_{\mathcal{A}}$ of the identity element $1_{\mathcal{A}} \in \mathcal{A}$:

$$\hat{1}_{\mathcal{A}} : (\omega, \rho) \mapsto | \, \omega \, \rangle \langle \, \omega \mid 1_{\mathcal{A}} \mid \rho \, \rangle \langle \, \rho \, |, \quad \forall p \in \hat{\mathcal{A}}, \ \forall \omega, \rho \in (\mathcal{P}_{\mathcal{A}})_p.$$

The gauge section is Hermitian: $\hat{1}_{\mathcal{A}}(\omega, \rho)^* = \hat{1}_{\mathcal{A}}(\rho, \omega)$.
In general it is not a subcategory of $\mathcal{E}_{\mathcal{A}}$.

Informally, the gauge section allows the specification of bundle gerbe acting as symmetry morphisms of a "horizontal categorified" site of "gauge blocks" inside the transition amplitudes bundle; a structure that is crucial in the reconstruction of a C*-algebra (isomorphic to $\mathcal{A}$) form $\mathcal{E}_{\mathcal{A}}$.

The Gauge Transition Amplitude Space

The gauge section $\hat{1}_{\mathcal{A}}$ induces on the set $(\mathcal{P}_{\mathcal{A}})_p$ a structure of **total Fell bundle-valued transition amplitude space**[28]:

- for all $\omega \in (\mathcal{P}_{\mathcal{A}})_p$, $\hat{1}_{\mathcal{A}}(\omega,\omega) = \mid \omega \rangle\langle \omega \mid = 1_{(\mathcal{E}_{\mathcal{A}})_{(\omega,\omega)}}$,
- $\hat{1}_{\mathcal{A}}(\omega,\rho)^* = \hat{1}_{\mathcal{A}}(\rho,\omega)$, for all $\omega,\rho \in (\mathcal{P}_{\mathcal{A}})_p$,
- there exists at least one **frame** i.e. a subset $\mathcal{F} \subset (\mathcal{P}_{\mathcal{A}})_p$ such that, for all $\omega,\rho \in (\mathcal{P}_{\mathcal{A}})_p$,
  $\hat{1}_{\mathcal{A}}(\omega,\rho) = \sum_{\theta\in\mathcal{F}} \hat{1}_{\mathcal{A}}(\omega,\theta) \circ \hat{1}_{\mathcal{A}}(\theta,\rho)$.

$\mathcal{I} \subset (\mathcal{P}_{\mathcal{A}})_p$ is **orthonormal** if $\omega \neq \rho \in \mathcal{F}, \Rightarrow \hat{1}_{\mathcal{A}}(\omega,\rho) = 0_{(\mathcal{E}_{\mathcal{A}})_{(\omega,\rho)}}$.

Frames are maximal othonormal.

The transition amplitude space is **total** if every maximal orthonormal set is a frame.

[28] For complex-valued transition amplitude spaces, see section 4.5 in: S.Gudder (1988) Quantum Probability, Elsevier.

The Gauge Bundle Gerbe

- Our gauge transition amplitude space is total, hence for every $p \in \hat{\mathcal{A}}$, **gauge frame blocks** coincide with maximal orthonormal sets of pure states i.e. sets $\mathcal{F} \subset (\mathcal{P}_{\mathcal{A}})_p$ such that the "matrix" $\hat{1}_{\mathcal{A}}|_{\mathcal{F}\times\mathcal{F}}$, for $\omega, \rho \in \mathcal{F}$, is the "identity":

$$\hat{1}_{\mathcal{A}}(\omega,\rho) = \begin{cases} |\,\omega\,\rangle\langle\,\omega\,| = 1_{(\mathcal{E}_{\mathcal{A}})_{(\omega,\omega)}}, & \omega = \rho, \\ 0_{(\mathcal{E}_{\mathcal{A}})_{(\omega,\rho)}}, & \omega \neq \rho. \end{cases}$$

- For two gauge frame blocks $\mathcal{F}_1, \mathcal{F}_2$, the "off-diagonal matrix" $\hat{1}_{\mathcal{A}}|_{\mathcal{F}_1\times\mathcal{F}_2}$ is a "unitary" with inverse $\hat{1}_{\mathcal{A}}|_{\mathcal{F}_2\times\mathcal{F}_1}$; hence we obtain a **gauge bundle gerbe** $\mathcal{F}_{\mathcal{A}}$ of gauge frame blocks over $\hat{\mathcal{A}}$.

The Spectral Bundle Gerbe

Restricting the transition amplitude bundle $\mathcal{E}_{\mathcal{A}}$ onto each one of the gauge blocks $\mathcal{F}_1 \times \mathcal{F}_2$ of the pair groupoid $\mathcal{F}_{\mathcal{A}}$ of gauge block frames we obtain a bundle gerbe enriched in the Morita 2-groupoid of full 1-C*-categories:

- Each bundle $\mathcal{E}_{\mathcal{A}}|_{\mathcal{F}\times\mathcal{F}}$ is a full 1-C*-category,
- The bundles $\mathcal{E}_{\mathcal{A}}|_{\mathcal{F}_1\times\mathcal{F}_2}$ are Morita isomorphism bimodules between the two C*-categories $\mathcal{E}_{\mathcal{A}}|_{\mathcal{F}_j\times\mathcal{F}_j}$, for $j = 1, 2$.

Possible links with Flori-Fritz's gleaves must be explored.[29]

[29] Compositories and Gleaves, arXiv:1308.6548.

The Transition Amplitude Uniformity

Every Fell bundle-valued transition amplitude space (P, γ) determines a uniformity on P: the family of entourages of Δ_P is given by the family of subsets given, for $0 < \epsilon < 1$, by:
$\mathcal{U}^\epsilon := \{(\omega, \rho) \in P \times P \mid \|\gamma(\omega, \rho)\| > 1 - \epsilon\} \subset P \times P$.

It will be a requirement for our spaceoids to assume that fibrewise the uniform structure of P coincides with the uniform structure induced by its Fell bundle-valued transition amplitude space structure. This requirement is always satisfied for spectral spaceoids and it is the counterpart, in our setting, of the coincidence between the weak*-uniformity and the Kähler metric uniformity in the phase-space of a quantum system.

The Enveloping C*-algebra of a C*-category

Given a C*-category $\mathcal{C}$, its **C*-envelope** is a unital C*-algebra $\Xi(\mathcal{C})$ with a $*$-functor $\iota : \mathcal{C} \to \Xi(\mathcal{C})$ that satisfies the following universal factorization property: for any other $*$-functor $\phi : \mathcal{C} \to \mathcal{A}$ into a unital C*-algebra $\mathcal{A}$, there exists a unique unital $*$-homomorphism $\hat{\phi} : \Xi(\mathcal{C}) \to \mathcal{A}$ such that $\phi = \hat{\phi} \circ \iota$.

Proposition

Every C-category $\mathcal{C}$ admits a C*-envelope $\mathcal{C} \xrightarrow{\iota} \Xi(\mathcal{C})$.*

We will denote by $\Xi^{**}(\mathcal{C})$ the **W*-envelope**.

- Non-commutative Spaceoids

Non-commutative Spaceoids 1

A **non-commutative spaceoid** $(E, \pi, P, \chi, X, \gamma)$ is a saturated unital Fell line-bundle $(E, \pi, P \times_X P)$ over a topological bundle of pair groupoids $(P \times_X P, \chi, X)$, where $\chi : P \to X$ is a surjective open continuous projection from a uniform Hausdorff space P onto the quotient space X; equipped with a continuous section $\gamma : P \times_X P \to E$ inducing on P a structure of Fell bundle-valued saturated full transition amplitude space that is compatible with the uniform topology of P.

Let us spell in detail the definition.

- P is a uniform space;
- the uniform completion of P is compact Hausdorff (topological saturation condition);

Non-commutative Spaceoids 2

- $P \xrightarrow{\chi} X$ is a surjective map onto the set X hence, when X is equipped with the quotient topology, the projection map χ is continuous and open; as a further consequence, when equipped with the restriction of the product topology $P \times_X P := \{(\omega, \rho) \in P \times P \mid \chi(\omega) = \chi(\rho)\} \subset P \times P$, is a topological groupoid, and so $P \times_X P \to X$ is a topological bundle of pair groupoids over X;
- $(E, \pi, P \times_X P)$ is a saturated unital Fell line-bundle over the topological groupoid $P \times_X P$;
- $P \times_X P \xrightarrow{\gamma} E$ is a continuous section of the previous bundle;
- (P, γ) is a Fell bundle-valued transition amplitude space that is total and algebraically saturated; Isbell reflective subcategory
- the uniform structure on P is fibrewise "compatible" with its uniform structure as a transition amplitude space (P, γ).

Morphisms of Non-commutative Spaceoids

A **morphism of non-commutative spaceoids** $(E^1, \pi^1, P^1, \chi^1, X^1, \gamma^1) \xrightarrow{(\lambda,\Lambda)} (E^2, \pi^2, P^2, \chi^2, X^2, \gamma^2)$ is given by a pair of maps $\lambda : P^1 \to P^2$ and $\Lambda : \lambda^\bullet(E^2) \to E^1$ such that:

- $\lambda : P^1 \to P^2$ is a uniformly continuous map such that $\chi^2 \circ \lambda = \chi^1$, hence it induces a necessarily continuous quotient map $[\lambda] : X^1 \to X^2$ and $(\lambda, \lambda) : P^1 \times_{X^1} P^1 \to P^2 \times_{X^2} P^2$ is necessarily a continuous homomorphism of bundles of pair groupoids;
- $\Lambda : \lambda^\bullet(E^2) \to E^1$ is a morphism of Fell bundles from the (λ, λ)-pull-back of E^2 to E^1;
- $(\lambda, \Lambda) : (P^1, \gamma^2) \to (P^2, \gamma^2)$ is a morphism of Fell bundle-valued transition amplitude spaces i.e. $\Lambda(\gamma^2(\lambda(p), \lambda(q))) = \gamma^1(p, q)$, for all $p, q \in P^1$.

The Category $\mathscr{S}$ of Non-commutative Spaceoids

We have a **category of non-commutative spaceoids** where composition of morphisms is defined as:

$$(\lambda', \Lambda') \circ (\lambda, \Lambda) := (\lambda' \circ \lambda,\ \Lambda \circ \lambda^\bullet(\Lambda') \circ \Theta^{E^3}_{\lambda',\lambda})$$

where $E^1 \xrightarrow{(\lambda,\Lambda)} E^2 \xrightarrow{(\lambda',\Lambda')} E^3$ and where $\Theta^{E^3}_{\lambda',\lambda} : (\lambda' \circ \lambda)^\bullet(E^3) \to \lambda^\bullet((\lambda')^\bullet(E^3))$ is the canonical isomorphism between standard pull-backs of bundles.

The Category $\mathscr{A}$ of Unital C*-algebras

For us a **morphism of unital C*-algebras** will be a $*$-homomorphism $\mathcal{A}_1 \xrightarrow{\phi} \mathcal{A}_2$ whose pull-back $\phi^\bullet : \mathcal{S}_{\mathcal{A}_2} \to \mathcal{S}_{\mathcal{A}_1}$ is pure state preserving: $\phi^\bullet(\mathcal{P}_{\mathcal{A}_2}) \subset \mathcal{P}_{\mathcal{A}_1}$.[30]
Unital $*$-epimorphism are a special case.

We have a category $\mathscr{A}$ of unital C*-algebras with the usual composition of such pure state preserving unital $*$-homomorphisms.

[30] This choice let us recover continuous maps in the usual commutative Gel'fand duality and simultaneously allows to limit the study to spaceoids defined using only pure states. It is possible to consider more general classes of morphism.

- Gel'fand-Naĭmark Duality

The Section Functor Γ

The **section functor** $\Gamma : \mathscr{S} \to \mathscr{A}$, associates to every non-commutative spaceoid $(E, \pi, P, \chi, X, \gamma)$ the unital C*-algebra $\Gamma(E)$ consisting of all the sections $\sigma : P \times_X P \to E$ such that:

- σ is continuous and its restriction $\sigma|_{\Delta_P}$ to the diagonal $\Delta_P := \{(p, p) \mid p \in P\}$ of P is uniformly continuous;[31]
- σ is gauge invariant i.e. for every pair of γ-orthonormal frames $F_1, F_2 \subset P$, $\sigma|_{F_1}$ and $\sigma|_{F_2}$ are related by $\sigma|_{F_2} = \mathrm{Ad}_{\gamma|_{F_2 \times F_1}}(\sigma|_{F_1}) = \gamma|_{F_2 \times F_1} \odot \sigma|_{F_1} \odot \gamma|^{\star}_{F_2 \times F_1}$, where $\odot$ and $\star$ denote here the convolution product and the adjoint involution in the C*-category with objects the maximal orthonormal γ-frames of P and 1-arrows the restrictions $\sigma|_{F_1 \times F_2}$.

[31] Note that since $E|_{\Delta_P}$ is trivial line-bundle equipped with a uniformity, this condition is perfectly defined.

The Algebraic Saturation Condition (tentative)

Given a point $o \in X$, let $\mathcal{F}_o$ be the family of γ-frames in the transition amplitude space $\chi_o^{-1} \subset P \times_X P$.

Consider for all possible choice functions $(U_F)_{F \in \mathcal{F}_o}$, where U_F is a unitary section of $E|_F$, the family $(U_F^\gamma)_{F \in \mathcal{F}_o}$ of their γ-orbits denoted by $U_F^\gamma \in \Xi_\gamma^{**}(E|_{\chi^{-1}(o)})$.

The spaceoid is **algebraically saturated** if, for all $o \in X$, there exists at least a choice map $(U_F)_{F \in \mathcal{F}_o}$ such that $(U_F^\gamma)_{F \in \mathcal{F}_o}$ is the family of all unitaries in $\Xi_\gamma^{**}(E|_{\chi^{-1}(o)})$.

This condition identifies the full reflective subcategory of the category of algebraic spaceoids that is in Isbell duality with the category of atomic W*-bundles.

back to spaceoids

The Section Functor Γ on Morphisms

The section functor Γ associates to a morphism of spaceoids $E^1 \xrightarrow{(\eta,\Omega)} E^2$ the usual pull-back of sections:
$\sigma \mapsto \Gamma_{(\eta,\Omega)}(\sigma) \in \Gamma(E^1)$, for all $\sigma \in \Gamma(E^2)$, where
$\Gamma_{(\eta,\Omega)}(\sigma) : (p, q) \mapsto \Omega(\sigma(\eta(p), \eta(q))) \in E^1$, for all $p, q \in P^1$.

We see that $\Gamma_{(\eta,\Omega)}$ is indeed well-defined and a unital $*$-homomorphism of unital C*-algebras that preserves pure states.

The functor $\Gamma : \mathscr{S} \to \mathscr{A}$ is contravariant.

The Spectrum Functor Σ

The **spectrum functor** $\Sigma : \mathscr{A} \to \mathscr{S}$ associates to every unital C*-algebra $\mathcal{A}$ its spectral non-commutative spaceoid $(\mathcal{E}_\mathcal{A}, \pi_\mathcal{A}, \mathcal{P}_\mathcal{A}, \chi_\mathcal{A}, \mathcal{X}_\mathcal{A}, \gamma_\mathcal{A})$, where

- $\mathcal{P}_\mathcal{A}$ is the family of pure states of $\mathcal{A}$ equipped with the weak*-uniformity,
- $\chi_\mathcal{A} : \mathcal{P}_\mathcal{A} \to \mathcal{X}_\mathcal{A} := \hat{\mathcal{A}}$ is the quotient map onto the usual spectrum of $\mathcal{A}$ (the family of equivalence classes of untarily equivalent representations),
- $(\mathcal{E}_\mathcal{A}, \pi_\mathcal{A}, \mathcal{P}_\mathcal{A} \times_{\mathcal{X}_\mathcal{A}} \mathcal{P}_\mathcal{A})$ is the transition amplitude Fell line-bundle of $\mathcal{A}$, as already constructed,
- $\gamma_\mathcal{A} := \hat{1}_\mathcal{A}$ is the Gel'fand transform of the identity element of $\mathcal{A}$ making $(\mathcal{P}_\mathcal{A}, \gamma_\mathcal{A})$ into a full Fell line bundle-valued transition amplitude space that is topologically and algebraically saturated.

The Spectrum Functor Σ on Morphisms

The spectrum functor associates to every unital $*$-epimorphism of unital C*-algebras $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ a morphism of non-commutative spaceoids $\Sigma(\mathcal{A}_2) \xrightarrow{(\lambda_\phi, \Lambda_\phi)} \Sigma(\mathcal{A}_1)$ where:

- $\lambda_\phi : \mathcal{P}_{\mathcal{A}_2} \to \mathcal{P}_{\mathcal{A}_1}$ is the usual ϕ-pull-back of pure states: $\lambda_\phi : \omega \mapsto \omega \circ \phi$, for all $\omega \in \mathcal{P}_{\mathcal{A}_2}$;[32]
- $\Lambda_\phi : \lambda_\phi^\bullet(\mathcal{E}_{\mathcal{A}_1}) \to \mathcal{E}_{\mathcal{A}_2}$ is the disjoint union, for $\omega, \rho \in \mathcal{P}_{\mathcal{A}_2}$, of the fiberwise linear relations $(\Lambda_\phi)_{(\omega,\rho)} \lambda_\phi^\bullet(\mathcal{E}_{\mathcal{A}_1})_{(\omega,\rho)} \to (\mathcal{E}_{\mathcal{A}_2})_{(\omega,\rho)}$, given for $x \in \mathcal{A}_1$ by:

$$| \lambda_\phi(\omega) \rangle\langle \lambda_\phi(\omega) \mid x \mid \lambda_\phi(\rho) \rangle\langle \lambda_\phi)(\rho) | \quad \mapsto \quad | \omega \rangle\langle \omega \mid \phi(x) \mid \rho \rangle\langle \rho | .$$

The spectrum functor $\Sigma : \mathscr{A} \to \mathscr{S}$ is contravariant.

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[32] The pure state preserving property of ϕ is necessary condition in order to define this map.

The Gel'fand Transform

The **non-commutative Gel'fand transform** $\mathfrak{G}_{\mathcal{A}} : \mathcal{A} \to \Gamma \circ \Sigma(\mathcal{A}) = \Gamma(\mathcal{E}_{\mathcal{A}})$ is simply the map that to every $x \in \mathcal{A}$ associates the $\gamma_{\mathcal{A}}$-invariant section $\hat{x} : \mathcal{P}_{\mathcal{A}} \times_{\mathcal{X}_{\mathcal{A}}} \mathcal{P}_{\mathcal{A}} \to \mathcal{E}_{\mathcal{A}}$ given by $\hat{x}(\omega, \rho) := \mid \omega \rangle \langle \omega \mid x \mid \rho \rangle \langle \rho \mid$, for all $\omega, \rho \in \mathcal{P}_{\mathcal{A}}$.

$\mathfrak{G}_{\mathcal{A}}$ is actually a unital $*$-homomorphism between C*-algebras.

The Gel'fand Isomorphism

Theorem

The Gel'fand transform $\mathfrak{G} : I_{\mathscr{A}} \to \Gamma \circ \Sigma$ *is a natural isomorphism.*

The Evaluation Transform

The **evaluation transform** $\mathfrak{E}_E : E \to \Sigma \circ \Gamma(E)$ from the spaceoid E to the spectral spaceoid of the unital C*-algebra $\Gamma(E)$ consists of a morphism of spaceoids $\mathfrak{E}_E := (\eta^E, \Omega^E)$ as follows:

- $\eta^E : P \to \mathcal{P}_{\Gamma(E)}$ associates to every $p \in P$ the map $\eta^E_p := \zeta^E_p \circ \mathrm{ev}^E_p : \Gamma(E) \to \mathbb{C}, \ \sigma \mapsto \zeta^E_p(\sigma(p,p))$ obtained composing the function $\mathrm{ev}^E_p : \Gamma(E) \to \Gamma(E)_{(p,p)}$ that evaluates every section $\sigma \in \Gamma(E)$ in the point $(p,p) \in P \times_X P$ with the canonical Gel'fand-Mazur isomorphism $\zeta^E_p : \Gamma(E)_{(p,p)} \to \mathbb{C}$ between one-dimensional unital C*-algebras;
- $\Omega^E : (\eta^E)^\bullet(\mathcal{E}_{\Gamma(E)}) \to E$ is the relation fiberwise defined as $\Omega^E_{(p,q)} : e \mapsto \sigma(p,q)$, for every $\sigma \in \Gamma(E)$ such that $\sigma(\eta^E(p), \eta^E(q)) = e$, where $p, q \in P$ and $e \in \Sigma \circ \Gamma(E)_{(\eta^E(p,q))} = (\eta^E)^\bullet(\mathcal{E}_{\Gamma(E)})_{(p,q)}$.

The Evaluation Isomorphism Theorem

Theorem

The evaluation transform is a natural isomorphism $I_{\mathscr{S}} \xrightarrow{\mathfrak{E}} \Sigma \circ \Gamma$.

Crucial ingredients here are the saturation conditions on the spaceoid. The proof makes use of the non-commutative Stone-Weierstrass theorem by J.Glimm:

A unital $*$-subalgebra $\mathcal{B}$ of a unital C*-algebra $\mathcal{A}$ that separates the states in the weak*-closure $\overline{\mathcal{P}}_{\mathcal{A}} \subset \mathcal{S}_{\mathcal{A}}$ of the set of pure states of $\mathcal{A}$ is norm dense in $\mathcal{A}$.

Commutative / Non-commutative Gel'fand Duality 1

We have the following pair of commutative diagrams of functors:

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{\Gamma} & \mathscr{A} \\ \mathfrak{F}\uparrow & & \uparrow\mathfrak{I} \\ \mathscr{T} & \xrightarrow[\Gamma^c]{} & \mathscr{C} \end{array} \qquad\qquad \begin{array}{ccc} \mathscr{S} & \xleftarrow{\Sigma} & \mathscr{A} \\ \mathfrak{F}\uparrow & & \uparrow\mathfrak{I} \\ \mathscr{T} & \xleftarrow[\Sigma^c]{} & \mathscr{C} \end{array}$$

where $\mathscr{T} \underset{\Sigma^c}{\overset{\Gamma^c}{\rightleftarrows}} \mathscr{C}$ is the commutative Gel'fand-Naĭmark duality.

$\mathfrak{I} : \mathscr{C} \to \mathscr{A}$ is the usual inclusion functor.

Commutative / Non-commutative Gel'fand Duality 2

$\mathfrak{F} : \mathscr{T} \to \mathscr{S}$ is the covariant functor that associates to every compact Hausdorff space T the topological spaceoid $(E^T, \pi^T, P^T, \chi^T, X^T, \gamma^T)$ with $X^T := T$, $P^T := T$, $\chi^T := \iota_T$ the identity map of T, $(E^T, \pi^T, P^T \times_{X^T} P^T)$ the trivial complex line-bundle on the diagonal $\Delta_T = T \times_T T$ of T and γ^T the identity constant section of E^T.[33]

$\mathfrak{F}$ associates to every continuous map $f : T_1 \to T_2$ of compact Hausdorff topological spaces the morphism of spaceoids $(\lambda^f, \Lambda^f) := (f, F)$, where F is the canonical isomorphism between trivial complex line-bundles $f^\bullet(E^{T_2}) \to E^{T_1}$ over T_1 (the f-pull-back of a trivial bundle is trivial).

[33] Notice that every compact Hausdorff space is equipped with a unique uniformity inducing its topology (enturages are the just neighborhoods of the diagonal of X) and continuous maps are uniform for such uniformity.

Commutative / Non-commutative Gel'fand Duality 3

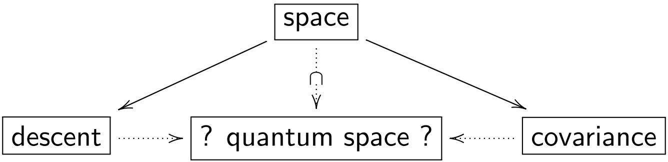
The following commutative diagrams between the commutative and non-commutative Gel'fand and evaluation transforms:

$$
\begin{array}{ccc}
\mathscr{S} & \xrightarrow{\mathfrak{E}^{\mathscr{S}}} & \mathscr{S} \\
\uparrow \mathfrak{F} & & \uparrow \mathfrak{F} \\
\mathscr{T} & \xrightarrow[\mathfrak{E}^{\mathscr{T}}]{} & \mathscr{C}
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{A} & \xrightarrow{\mathfrak{G}^{\mathscr{A}}} & \mathscr{A} \\
\uparrow \mathfrak{I} & & \uparrow \mathfrak{I} \\
\mathscr{C} & \xrightarrow[\mathfrak{G}^{\mathscr{C}}]{} & \mathscr{C}
\end{array}
$$

finally prove that our non-commutative Gel'fand-Naĭmark duality is a natural extension of the usual commutative case (and reduces to it by restricting to the embedded full sub-categories $\mathfrak{F}(\mathscr{T}) \subset \mathscr{S}$ and $\mathfrak{I}(\mathscr{C}) \subset \mathscr{A}$).

- What Are the "Spectra" of C*-algebras?

The Two "Souls" of Geometry



Descent theory (gluing)	Covariance (transport)
sheaves/stacks	Klein-Cartan geometries
Grothendieck topoi	Ehresmann connections
Grothendieck categories	category theory
	(higher) homotopy/holonomy

In a quantum space:

(a) relations between points are primary concepts,

(b) transport depends on a transition amplitude structure.

Non-commutative Klein-Cartan Geometries?

Klein's Erlangen program characterizes geometry from its group of symmetries: *Klein's geometries are homogeneous spaces.*[34]

Cartan dealt with local symmetries: *Cartan's geometries are bundles of homogeneus spaces with a connection.*[35]

We need an understanding of non-commutative covariance and transport (and how they merge with the descent data of the space).

[34] F.Klein (1872) arXiv:0807.3161.

[35] See the book: R.W.Sharpe (1997) Differential Geometry: Cartan's Generalization of Klein's Erlangen Program, Springer.

Non-commutative Topoi / Descent Theory?

We probably need a version **non-commutative topos theory** where categories of tensor products and involutions of bimodules take the place of the usual Cartesian closed categories environment.

- P.Cartier[36] (see also[37])
- F. van Oystaeyen[38] (see also[39])
- M.Kontsevich, A.Rosenberg (NC descent theory)[40]
- C.Flori, T.Fritz (Gleaves)[41]

[36] A Mad Day's Work: from Grothendieck to Connes and Kontsevich: The Evolution of Concepts of Space and Symmetry (2001) Bull Amer Math Soc 38:389-408.

[37] T.Maszczyk, `arXiv:math/0611806`.

[38] Virtual Topology and Functor Geometry (2007) CRC.

[39] K.Cvetko-Vah, J.Hemelaer, L.Le Bruyn `arXiv:1705.02831`.

[40] arXiv:math/9812158; see also S.Mahanta arXiv:math/0501166.

[41] `arXiv:1308.6548`.

Thank You for Your Kind Attention!

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