Banach Spaces and Linear Operators
Part I

MUIC Seminar

September 21, 2016
Banach Spaces

Definition (Banach Space)

Let \((X, \| \cdot \|)\) be a vector space (possibly infinite-dimensional) and \(\| \cdot \| : X \to \mathbb{R}_+\).

- The space \(X\) is said to be a normed space if
  - \(\| x \| = 0 \iff x = 0\),
  - \(\| \lambda x \| = |\lambda| \| x \|\) for \(\lambda \in \mathbb{R}(\mathbb{C})\),
  - \(\| x + y \| \leq \| x \| + \| y \|\).

The function \(\| \cdot \|\) is called a norm. \((X, \| \|)\) is a metric space under the metric \(d(x, y) = \| x - y \|, \, x, y \in X\).

The function \(\| \cdot \| : X \to \mathbb{R}\) determines a metric on \(X\) as follows

\[
d(x, y) := \| x - y \| \quad \text{for } x, y \in X.
\]

It is easy to see that so defined function \(d\) is, indeed, a (translation invariant) metric on \(X\), i.e. \((X, d)\) is a metric (and, therefore, topological) space.
STEFAN BANACH [1892–1945]
Let \( x \in X \) and \( \epsilon > 0 \). The set \( B(x, \epsilon) = \{ y \in X : \| x - y \| \leq \epsilon \} \) is called a ball (centered about \( x \) and of radius \( \epsilon \)). The set of all balls \( B(x, \epsilon), \epsilon > 0 \) is a neighbourhood basis of the point \( x \). Due to the translation invariance of the metric \( d \) associated with the norm \( \| \cdot \| \), it is easy to see that \( B(x, \epsilon) = x + B(0, \epsilon) \), and so the neighbourhood basis of the origin (i.e. vector 0) determines the neighbourhood basis of every point \( x \in X \).

Additionally, since the topology of \( X \) is determined by the metric, we can select a countable set of \( \epsilon \)'s, e.g. \( \epsilon_n = n^{-1}, n = 1, 2, \ldots \) to determine the neighbourhood basis of the origin.
Given the properties (2) and (3) of the norm, we can easily see that if \( x, y \in B(0, \epsilon) \) and \( 0 \leq \lambda \leq 1 \) then

\[
\| \lambda x + (1 - \lambda) y \| \leq \lambda \| x \| + (1 - \lambda) \| y \| \leq \lambda \epsilon + (1 - \lambda) \epsilon = \epsilon,
\]

i.e. \( \lambda x + (1 - \lambda) y \) \( \in B(0, \epsilon) \) or, otherwise, \( B(0, \epsilon) \) is a convex set.

This can be expressed as

**Theorem**

A normed space \((X, \| \cdot \|)\) is a locally convex topological space under its natural metrizable topology derived from the norm. This topology is generated by the base of neighbourhoods of the origin \( B_n = \{ x \in X : \| x \| \leq n^{-1} \} \).
The concept of a Banach Space is one of the most important concepts of modern Functional Analysis. It originated from the XIX century developments in the Mathematical Analysis and it was given its modern form in the 1920’s, mainly through the work of Stefan Banach and his collaborators in Lwów, Poland (Hugo Steinhaus, Stanisław Mazur, Władysław Orlicz, and Juliusz Schauder, to name a few). The spaces were called Banach Spaces in honour of Stefan Banach but, of course, he himself did not use this term. The name Banach Spaces came in general use after 1945 when Banach already died. The theory of what is now known as Banach Space Theory was given its solid form in the 1932 with the appearance of the book by Banach himself, *Théorie des opérations linéaires*. Despite the passage of 80 years of intensive research and tremendous progress in all areas discussed in the book, this monograph remains extremely useful. In fact, many questions raised in the book were solved only very recently or remain unanswered.
Banach Spaces are complete normed spaces. The concept of completeness is one of the most useful concepts of Analysis. Whenever we have an infinite sequence, we can only inspect a finite number of terms. If the space is complete, we are assured that what happens with the very last terms of the sequence is similar to the behaviour of the initial terms, i.e. the limit behaves similarly to the rest of the sequence (provided that the terms show some tendency towards convergence, as expressed by the Cauchy Condition). Thus does not mean, of course, that the incomplete normed spaces are uninteresting. In fact, as we shall see later, some of them are quite useful.
Classical Banach Spaces

- $\mathbb{R}^n$: the space of all (real) $n$–sequences $x = (x_1, x_2, \ldots, x_n)$ with the Euclidean norm $\|x\| = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$. It is known from elementary real analysis that this space is a Banach space.

- $P(0, 1)$ – the space of all polynomials on the interval $[0, 1]$. The norm in the space is the so–called supremum norm:

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in [0, 1]\}$$

The space $(P(0, 1), \|\cdot\|_\infty)$ is not complete, i.e. it is not a Banach space. In fact, the sequence of polynomials

$$x_n(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}, \quad n = 1, 2, \ldots$$

satisfies the Cauchy condition but $x_n(t) \to e^t$ as $n \to \infty$ and $e^t$ is not a polynomial.
- $c_0$ – the space of all sequences of real numbers which converge to 0. In other words, $x \in c_0$ if $x = (x_1, x_2, x_3, \ldots)$ and $x_n \to 0$ ($n \to \infty$). This space, equipped with the sup norm: $\|x\|_\infty = \sup_{n \in \mathbb{N}}|x_n|$, is a Banach space.

- $c$ – the space of all convergent sequences with the sup norm (as in the case of $c_0$ above) is a Banach space.

- $\ell_\infty$ – the space of all bounded sequences. This space is also a Banach space under the sup norm.
- $\ell_p$, $1 \leq p < \infty$ – the spaces of all $p$–summable sequences. Here the norm is

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p},$$

and $(\ell_p, \|x\|_p)$ is a Banach space for every $1 \leq p < \infty$.

In order to prove that $\ell_p$’s are vector spaces and normed spaces one has to use the Minkowski and Hölder inequalities. We shall prove the Hölder inequality separately. Also note that the space $\ell_2$ is an infinite dimensional version of the Euclidean $n$–space of Example 1. This is an important space, it is called a Hilbert Space and it was perhaps the first example of a vector space thoroughly investigated as an object in Functional Analysis. It also is the "best" Banach space, meaning that it has the most regular properties of all spaces.
- $C(0, 1)$ – the space of all continuous functions on the interval $[0, 1]$, with the sup norm
  \[
  \|x\|_\infty = \sup\{|x(t) : t \in [0, 1]|, \]

  is a Banach space.

- The spaces $L_p(0, 1), 1 \leq p < \infty$ of $p$–power Lebesgue integrable functions on $[0, 1]$ with the norm
  \[
  \|x\| = \left( \int_0^1 |x(t)|^p dt \right)^{1/p}.
  \]

  Here we identify functions functions that differ only on a set of zero measure.

- $L_\infty(0, 1)$ – the space of all (essentially) bounded functions on $[0, 1]$ with the norm
  \[
  \|x\| = \text{esssup}\{|x(t)| : t \in [0, 1]|.
  \]

  $L_\infty(0, 1)$ is a Banach space.
**Definition**

Let $X$ and $Y$ be two normed spaces. A function $T : X \to Y$ is called a linear operator if

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y),$$

for all $x, y \in X$ and $\lambda, \mu \in \mathbb{R}$.

A linear operator is continuous if $x_n \to x \Rightarrow T(x_n) \to T(x)$ for every convergent sequence $\{x_n\}$ in $X$.

Clearly, due to linear nature of $X$, $Y$ and $T$, a linear operator is continuous if and only if it is continuous at 0, i.e. $T(x_n) \to 0$ whenever $x_n \to 0$. Here $x_n \to 0$ means that $\|x_n\| \to 0$ and $T(x_n) \to 0$ means that $\|T(x_n)\| \to 0$. 
A linear operator \( T : X \rightarrow Y \) is bounded if there exists a constant \( M > 0 \) such that \( \| T(x) \| \leq M \| x \| \) for all \( x \in X \). This implies several things. First that \( T \) is bounded on a unit ball of \( X \), i.e. the set \( B(0, 1) = \{ x \in X : \| x \| \leq 1 \} \). That’s why the operator is called bounded.

Secondly, it follows immediately that a bounded linear operator is continuous. It is less obvious but equally easy to show that a linear operator which is continuous is also bounded.
Theorem

The Following Are Equivalent for a linear operator $T$ between two normed spaces.

- $T$ is continuous ,
- $T$ is continuous at a point $x \in X$ ,
- $T$ is continuous at 0 ,
- $T$ is bounded .

The set of all bounded operators between two normed spaces is typically denoted by $\mathcal{L}(X, Y)$.

Let $T \in \mathcal{L}(X, Y)$. Then, since $T$ is bounded, there exists a constant $M > 0$ such that $\|T(x)\| \leq M\|x\|$ for all $x \in X$. Set $\|T\| := \inf\{M \geq 0 : M\|x\| \geq \|T(x)\|\}$. (The infimum exists; the set of $M$'s is bounded below by 0). The number $\|T\|$ associated with $T$ is an operator norm (or simply a norm) of an operator $T$. 


Therefore \((\mathcal{L}(X, Y), \| \cdot \|)\) is a normed space. If \(Y\) is a Banach space it is easy to show that \((\mathcal{L}(X, Y), \| \cdot \|)\) is a Banach space.
**Definition**

Let $X$ be a Banach space. A sequence $\{e_n\}$ of elements of $X$ is called a Schauder basis (or, simply, a basis) if every element of $X$ can be uniquely represented as a series $\sum_{n=1}^{\infty} a_n e_n$, where $a_n \in \mathbb{R}$, $n = 1, 2, \ldots$.

Not every Banach space has a basis. It is easy to see from the definition that if a space $X$ has a basis it must be separable. It is so because $\overline{\text{lin}}\{x_n\} = X$ and $\{x_n\}$ is, of course, countable. But most separable Banach spaces have bases. In fact, the spaces $c_0, \ell^p$, where $1 \leq p < \infty$ have the simplest basis: the set of all unit vectors $\{\delta_{i,j}\}$, $i, j = 1, 2, \ldots$. A more complicated bases also exist in
$L_p(0, 1), 1 \leq p < \infty$: the Haar system. The space $C(0, 1)$ also has a basis. The spaces $\ell_\infty$ and $L_\infty(0, 1))$ are nonseparable, and as such, do not have a basis.

It was for a long time an open problem whether every separable Banach space has a basis. It was only in 1973 that Per Enflo showed that there exists a separable Banach space without a basis. Today we know more spaces of this kind, some of them quite natural.
Enflo receiving a goose from Mazur
Enflo receiving a goose from Mazur
It was already known to Banach that every Banach space (separable or not) has a weaker property.

**Definition**

A Banach space $X$ has a basic sequence if there exists a sequence $\{e_n\}$ of elements of $X$ such that $\{e_n\}$ is a basis for its closed linear span.

**Theorem (Banach)**

Every infinite – dimensional Banach space contains a basic sequence.
Definition

A linear operator $T : X \to Y$ is called a compact operator, if the image of the unit ball $B_X$ of $X$ is a precompact set in $Y$.

Definition

A sequence $\{x_n\}_{n=1}^{\infty}$ is called an unconditional basic sequence if it is a basic sequence for all permutations $\pi(\mathbb{N})$ of the index set $\{1, 2, 3, \ldots\}$.

Problem

Does every Banach Space have an unconditional basic sequence?

Theorem (Gowers–Maurey, 1993)

There exists a Banach Space $\mathfrak{x}_{gm}$ without an unconditional basic sequence.
TIMOTHY GOWERS – Fields Medallist, 1994
ALEXANDRE GROTHENDIECK (1928 – 2014) – Fields Medallist, 1966
Nuclear Spaces

ALEXANDRE GROTHENDIECK (1928 – 2014) – Fields Medallist, 1966
Nuclear Spaces, II
Nuclear Spaces, II