Spaces With Small Operators

d'après

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A Herdeditarily Indecomposable $\mathscr{L}_\infty$–space
that solves the scalar–plus–compact problem

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Unusual Banach Spaces I

Theorem (James,1950)

Let X be a Banach space with an unconditional basis. Then X is reflexive if and only if neither ℓ_1 nor c_0 is isomorphic to a subspace of X.

Theorem (Tsirelson,1974)

There is a reflexive and separable Banach space T that contains no isomorphic copy of either c_0 or ℓ_p, $1 \leq p < \infty$.

Unusual Banach Spaces II

The idea behind the construction of Tsirelson's is as follows. Take the c_{00} space *i.e.* the space of all finite sequences (or the space $c_{00}(\Gamma)$ – the space of functions with finite support) and construct a norm $\|\cdot\|_T$ so that the completion $T = \overline{c_{00}}^{\|\cdot\|_T}$ contains neither ℓ_p nor c_0. The basic tool is to see that the new norm $\|\cdot\|$ is such that the mapping $P_{X_0} : T \to X_0$ is not continuous (where X_0 is a subspace of T algebraically isomorphic to, say, ℓ_1) . That is, the new norm $\|\cdot\|_T$ is so selected that $\|x\|_T \leq C \cdot \sum_{\gamma\in\Gamma} |x(\gamma)|$ is not satisfied for any constant C. Clearly, no such norm can be defined globally. It is through meticulous construction of the norm on subspaces and then defining the global norm in inductive manner can such norm be shown to exist. The actual construction of Tsirelson's, while simple in concept, is the height of mathematical virtuosity.

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Professor T showing his students how simple is the construction of the Tsirelson Space.

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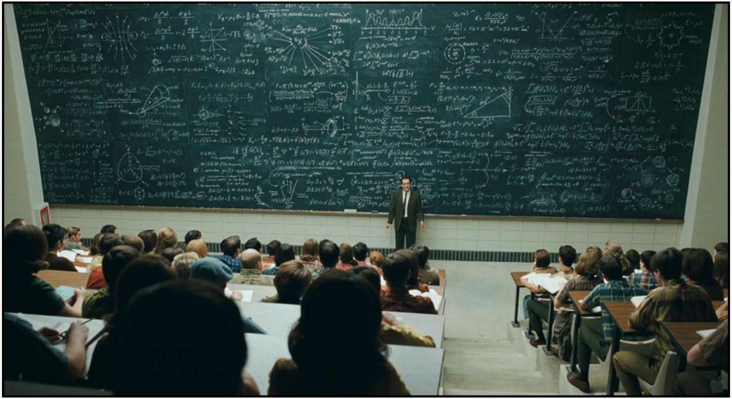

Definition (Distortable Space)

Let $(X, \|\cdot\|)$ be an infinite–dimensional Banach space and let $\lambda > 1$.

- The space X is said to be λ–distortable if there exists an equivalent norm $|\cdot|$ on X such that, for every infinite–dimensional subspace Y of X

$$sup \left\{ \frac{|y_1|}{|y_2|} : \|y_1\| = \|y_2\| = 1, y_1, y_2 \in Y \right\} > \lambda.$$

- X is said to be distortable (resp. arbitrarily distortable) if X is λ–distortable for some (resp. all) $\lambda > 1$.

Theorem (Milman, 1969/71)

If X is not distortable then X contains a copy of c_0 or ℓ_p for some $1 \leq p < \infty$.

Corollary

Tsirelson space T is distortable.

In 1991, T. Schlumprecht produced the famous example of an arbitrarily distortable ”Tsirelson – like” Banach space S. The refinement of construction of this type of Banach space led to the Gowers – Maurey construction of the Banach space with no unconditional basic sequence

Theorem (Schlumprecht, 1991)

There exists a Banach space S (the Schlumprecht space) which is arbitrarily distortable.

Theorem (Gowers, 1994)

There exists a Banach space $\mathfrak{X}_{gm}$ not containing c_0, ℓ_1 or a reflexive subspace.

An Old Problem

Problem

Does there exist a Banach space X such that every $T \in \mathcal{L}(X, X)$ is of the form $T = \lambda I + K$, where $\lambda \in \mathbb{R}$ and $K \in \mathscr{K}(X, X)$ is a compact operator.

This problem was first stated explicitely by J. Lindenstrauss in 1976 but its roots are in the early paper of N. Aronszajn and K. T. Smith in 1954.

The discovery of a pathological Banach space $\mathfrak{X}_{g,m}$ by T. Gowers and B. Maurey in 1993 and subsequent work by T. Gowers along this lines shed a new light onto the problem.

Strictly Singular Operators

Definition

An operator $T : X \to X$ is called strictly singular if for every closed subspace Y of X such that $T_{|Y}$ is an isomorphism, we have $dim\, Y < +\infty$.

Definition

- A Banach space X is indecomposable if there do not exist infinite dimensional closed subspaces Y and Z of X with $X = Y \oplus Z$;

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- A Banach space X is indecomposable if there do not exist infinite dimensional closed subspaces Y and Z of X with $X = Y \oplus Z$;
- A Banach space X is Hereditarily Indecomposable (HI space) if every closed subspace of X is indecomposable.

Theorem (Gowers–Maurey, 1993)

There exists an HI space.

Note

- The HI space of the Theorem is usually denoted $\mathfrak{X}_{gm}$.

Theorem (Gowers (1999), Gasparis (2003))

Every operator $T : \mathfrak{X}_{gm} \to \mathfrak{X}_{gm}$ is of the form $T = \lambda I + S$, where S is a strictly singular operator.

Theorem

Let X and Y be Banach spaces. Every compact operator $T \in \mathcal{L}(X, Y)$ is strictly singular but there exist operators which are strictly singular but not compact.

Theorem

An operator between Hilbert spaces is compact if and only if it is strictly singular.

Example

If $1 \leq p < r < \infty$, then the natural embedding $J : \ell_p \to \ell_r$ is a non–compact strictly singular operator.

Every operator T on the HI space $\mathfrak{X}_{gm}$ is a strictly singular perturbation of the identity but not a compact perturbation of the identity (*i.e.* is not a solution to the question of Lindenstrauss), as was shown by G. Androulakis and Th. Schlumprecht in 2001.

Theorem (Androulakis–Schlumprecht, 2001)

There exists a strictly singular, non–compact operator on the space $\mathfrak{X}_{gm}$.

Basic properties of Fredholm Operators

Definition

- An operator $T \in \mathcal{L}(X, Y)$ is a Fredholm Operator if $T(X)$ is closed and $ker\ T$ and $\mathrm{coker}\ T = Y/T(X)$ are finite–dimensional.
- $n(T) :=\dim ker\ T$, $d(T) := \dim \mathrm{coker}\ T$.
- index of T $\quad i(T) := n(T) - d(T)$

Theorem

- *A bounded linear operator is Fredholm if and only if both $n(T)$ and $d(T)$ are finite;*
- *A bounded linear operator $T : X \to Y$ between Banach spaces is Fredholm if and only if the operator $T_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is Fredholm. In this case $i(T_{\mathbb{C}}) = i(T)$;*
- *If T is a Fredholm operator with index i and S is strictly singular then $T + S$ is Fredholm with index i.*

Properties of the $\mathfrak{X}_{gm}$ space

(A) A non–trivial projection P (*i.e.* PX and $(I-P)X$ are both infinite–dimensional) is not of the form $\lambda I + S$;

(B) If T is Fredholm with index i and S is strictly singular then $T + S$ is Fredholm with index i;

(C) From **(A)** $+$ **(B)** $\Rightarrow X \neq Y \oplus Z$ with both Y and Z infinite–dimensional ;

(D) $X \ncong Y$ for any proper infinite–dimensional subspace Y of X, for any isomorphism $X \to Y$ is Fredholm with a non–zero index, while $\lambda I + S$ has zero index .

Definition

- A separable Banach space X is an $\mathscr{L}_{\infty,\lambda}$–space if there is and increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite dimensional subspaces of X such that the union $\bigcup_{n\in\mathbb{N}} F_n$ is dense in X and, for each n, F_n is λ–isomorphic to $\ell_\infty^{\dim F_n}$;
- λ–isomorphic $\Leftrightarrow \exists$ isomorphism $\mu : F_n \to \ell_\infty^{\dim F_n}$ with $\|\mu\| \cdot \|\mu^{-1}\| < \lambda$;
- We say that X is an $\mathscr{L}_\infty$–space if it is an $\mathscr{L}_{\infty,\lambda}$–space for some $\lambda > 1$.

Theorem (Lewis–Stegall, 1971)

If a separable $\mathscr{L}_\infty$–space X has no subspace isomorphic to ℓ_1 then its dual X^* is isomorphic to ℓ_1.

- This implies that the dual of a separable HI $\mathscr{L}_\infty$–space is isomorphic to ℓ_1;
- Bourgain and Delbaen (1980) constructed an $\mathscr{L}_\infty$–space $X_{a,b}$ which does not contain c_0.

Theorem (Spiros A. Argyros and Richard G. Haydon)

There exists a hereditarily indecomposable $\mathscr{L}_\infty$–space $\mathfrak{X}_K$ with dual isomorphic to ℓ_1 such that every bounded linear operator T on this space is of the form $T = \lambda I + K$, where $K \in \mathscr{K}(X)$.

Proof.

The space $\mathfrak{X}_K$of the theorem is a modification of $\mathfrak{X}_{gm}$ with specific design to be an $\mathscr{L}_\infty$–space so that its dual is isomorphic to ℓ_1. The fact that it has only compact perturbations of identity as operators was, according to the authors, a ”lucky shot”. □

Definition (Invariant Subspace)

Let $T : X \to X$ be a bounded linear operator on a Banach space X and let V be a proper subspace of X (*i.e.* $\emptyset \neq V \neq X$). We say that V is invariant under T if $T(V) \subseteq V$.

Problem

When does a bounded linear operator on an infinite dimensional Banach space have a non–trivial closed invariant subspace ?

- P. Enflö (1987) showed an example of a bounded linear operator on a real or complex separable non–reflexive space Banach space without non–trivial invariant subspaces;
- C. J. Read (1986/87) showed an example of a bounded linear operator on the space ℓ_1 without non–trivial closed invariant subspaces.

Lomonosov Theorem I

Theorem (Lomonosov Theorem)

If a bounded linear operator T on an infinite dimensional real or complex Banach space commutes with a non–zero compact operator then T has a non–trivial closed invariant subspace.

Corollary

Every operator T on the space $\mathfrak{X}_K$ of Argyros and Haydon has a non–trivial invariant subspace.

Lomonosov Theorem II

Proof.

$$TK = (\lambda I + K)K = \lambda K + K^2 = K(\lambda I + K) = KT$$

for every $K \in \mathscr{K}(X)$, $K \neq 0$. By the Lomonosov Theorem T has a non–trivial invariant subspace. If $K = 0$ then T trivially commutes with any non–zero compact operator. □

Note

The space $\mathfrak{X}_K$ is the first (and so far only known) infinite–dimensional Banach space such that every operator on it has a non–trivial invariant subspace.

Finite Rank Operators

Definition (Finite Rank Operator)

- Let X, Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$. The mapping T is called a finite rank operator if its image space is finite dimensional.
- Each finite rank mapping T can be represented in the form

$$Tx = \sum_{i=1}^{n} \langle x, x_i^* \rangle y_i$$

  for $x \in X$ with $y_i \in Y$ and $x_i \in X$ for $i = 1, \ldots, n$.
- The collection of all finite rank operators from X to Y is a linear subspace of $\mathcal{L}(X, Y)$ and is denoted by $\mathscr{A}(X, Y)$.

Nuclear Operators I

Definition (Nuclear Operator)

Let X, Y be normed spaces and let U, V be closed unit balls in X *resp.* Y. Let p_V and p_{U^*} be Minkowski functionals of V *resp.* U^*. An operator $T \in \mathcal{L}(X, Y)$ is a nuclear operator if there exist sequences $x_n^* \in X^*$ and $y_n \in Y$, $(n \in \mathbb{N})$ with

$$\sum_{n\in\mathbb{N}} p_{U^*}(x_n^*) p_V(y_n) < \infty$$

such that

$$Tx = \sum_{n\in\mathbb{N}} \langle x, x_n^* \rangle y_n \text{ for } x \in X\,.$$

Nuclear Operators II

Definition

For each nuclear mapping $T : X \to Y$ we set

$$\nu(T) := \inf \left\{ \sum_{n \in \mathbb{N}} p_{u^*}(x_n^*) p_v(y_n) \right\}$$

where the infimum is taken over all possible representations of T.
The function ν is a norm on $\mathscr{N}(X, Y)$.

Theorem

For a normed space X and a Banach space Y the space $(\mathscr{N}(X, Y), \nu)$ is a Banach space.

Nuclear Operators III

Theorem

The space of all finite rank operators $\mathscr{A}(X, Y)$ is dense in the space of all nuclear operators $\mathscr{N}(X, Y)$.

Natural Operators on Banach Spaces

Natural Operators

The following types of operators $T : X \to X$ exist on all Banach spaces X :

- (Multiple of) identity λI;
- Finite rank operators $\mathscr{A}(X)$;
- Nuclear operators $\mathscr{N}(X)$.

Problem (1)

Does every operator on a Hilbert space have a non–trivial invariant subspace ?

Problem (2)

Does there exist an infinite–dimensional Banach X space such that every operator $T \in \mathcal{L}(X)$ is of the form $T = \lambda I + N$, where $N \in \mathscr{N}(X)$?

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