BANACH SPACES – THE INTRODUCTION

PRZEMO T. KRANZ

1. DEFINITIONS, EXAMPLES, and ELEMENTARY PROPERTIES.

A vector space $(X, +, \cdot)$ is called a *normed space* if there is a real valued function $\|\cdot\| : X \to \mathbb{R}_+$, called a *norm* such that

(1) $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$,

(3) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

A normed space will be denoted by $(X, \|\cdot\|)$, where $\|\cdot\|$ is a norm on X.

NOTE : We consider here only real vector spaces, *i.e.* such that the scalar field $\mathfrak{S}$ is of real numbers (and so the norm is a real valued function). It is possible (and quite interesting) to investigate other normed spaces *e.g.* complex normed spaces, or generally, normed spaces over general Archimedean or non–Archimedean fields.

The function $\|\cdot\| : X \to \mathbb{R}$ determines a *metric* on X as follows

$$d(x, y) := \|x - y\| \quad \text{for } x, y \in X .$$

It is easy to see that so defined function d is, indeed, a (translation invariant) metric on X, *i.e.* (X, d) is a *metric* (and, therefore, *topological*) space .

Let $x \in X$ and $\epsilon > 0$. The set $B(x, \epsilon) = \{y \in X : \|x - y\| \leq \epsilon\}$ is called a *ball* (centered about x and of radius ϵ). The set of all balls $B(x, \epsilon), \epsilon > 0$ is a *neighbourhood basis* of the point x. Due to the translation invariance of the metric d associated with the norm $\|\cdot\|$, it is easy to see that $B(x, \epsilon) = x + B(0, \epsilon)$, and so the neighbourhood basis of the *origin* (*i.e.* vector 0) determines the neighbourhood basis of every point $x \in X$.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Additionally, since the topology of X is determined by the metric, we can select a countable set of ϵ's, *e.g.* $\epsilon_n = n^{-1}, n = 1, 2, \dots$ to determine the neighbourhood basis of the origin.

Given the properties (2) and (3) of the norm , we can easily see that if $x, y \in B(0, \epsilon)$ and $0 \le \lambda \le 1$ then

$$\|\lambda x + (1 - \lambda)y\| \le \lambda\|x\| + (1 - \lambda)\|y\| \le \lambda\epsilon + (1 - \lambda)\epsilon = \epsilon\,,$$

i.e. $\lambda x + (1 - \lambda y) \in B(0, \epsilon)$ or, otherwise, $B(0, \epsilon)$ is a *convex* set.

This can be expressed as

THEOREM *A normed space* $(X, \|\cdot\|)$ *is a locally convex topological space under its natural metrizable topology derived from the norm. This topology is generated by the base of neighbourhoods of the origin* $B_n = \{x \in X : \|x\| \le n^{-1}\}$.

DEFINITION Let $(X, \|\cdot\|)$ be a normed space. If (X, d) is a complete metric space then $(X, \|\cdot\|)$ is called a *Banach space.*

Clearly, a normed space is a Banach space if every Cauchy sequence $\{x_n\}$ in X converges in X. This means, of course, that if $\{x_n\}$ is a sequence in X then

$$(\star) \qquad \forall_{\epsilon>0}\exists_{N_\epsilon}\forall_{m,n\ge N_\epsilon}\|x_n - x_m\| < \epsilon \Rightarrow \lim_{n\to\infty} x_n = x_0 \in X\,.$$

The concept of a Banach Space is one of the most important concepts of modern Functional Analysis. It originated from the XIX century developments in the Mathematical Analysis and it was given its modern form in the 1920's, mainly through the work of Stefan Banach and his collaborators in Lwów, Poland (Hugo Steinhaus, Stanisław Mazur, Władysław Orlicz, and Juliusz Schauder, to name a few). The spaces were called Banach Spaces in honour of Stefan Banach but, of course, he himself did not use this term. The name Banach Spaces came in general use after 1945 when Banach already died. The theory of what is now known as Banach Space Theory was given its solid form in the 1932 with the appearance of the book by Banach himself, *Théorie des opérations linéaires.* Despite the passage of 70 years of intensive research and tremendous progress in all areas discussed in the book, this monograph remains extermely useful. In fact, many questions raised in the book were solved only very recently or remain unanswered.

Banach Spaces are complete normed spaces. The concept of completeness is one of the most useful concepts of Analysis. Whenever we have an infinite sequence, we can only inspect a finite number of terms. If the space is complete, we are assured that what happens with the very last terms of the sequence is similar to the behaviour of the initial terms, *i.e.* the limit behaves similarly to the rest of the sequence (provided that the terms show some tendency towards convergence, as expressed by the condition ($\star$)). Thus does not mean, of course, that the incomplete normed spaces are uninteresting. In fact, as we shall see later, some of them are quite useful.

CLASSICAL BANACH AND NORMED SPACES.

1. $\mathbb{R}^n$: the space of all (real) n–sequences $x = (x_1, x_2, \ldots, x_n)$ with the Euclidean norm $\|x\| = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$.

It is known from elementary real analysis that this space is a Banach space.

2. $P(0,1)$ – the space of all polynomials on the interval $[0,1]$. The norm in the space is the so–called *supremum* norm :

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in [0,1]\}$$

The space $(P(0,1), \|\cdot\|_\infty)$ is not complete, *i.e.* it is not a Banach space. In fact, the sequence of polynomials

$$x_n(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}, n = 1, 2, \ldots$$

satisfies the Cauchy condition but $x_n(t) \to e^t$ as $n \to \infty$ and e^t is <u>not</u> a polynomial.

3. c_0 – the space of all sequences of real numbers which converge to 0. In other words, $x \in c_0$ if $x = (x_1, x_2, x_3, \ldots)$ and $x_n \to 0 \quad (n \to \infty)$. This space, equipped with the sup norm : $\|x\|_\infty = \sup_{n\in\mathbb{N}}|x_n|$, is a Banach space.

4. c – the space of all convergent sequences with the sup norm (as in the case of c_0 above) is a Banach space.

5. ℓ_∞ – the space of all bounded sequences. This space is also a Banach space under the sup norm.

6. $\ell_p, 1 \le p < \infty$ – the spaces of all p–summable sequences. Here the norm is

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}},$$

and $(\ell_p, \|x\|_p)$ is a Banach space for every $1 \le p < \infty$.

In order to prove that ℓ_p's are vector spaces and normed spaces one has to use the Minkowski and Hölder inequalities. We shall prove the Hölder inequality separately. Also note that the space ℓ_2 is an infinite dimensional version of the Euclidean n–space of Example 1. This is an important space, it is called a *Hilbert Space* and it was perhaps the first example of a vector space thoroughly investigated as an object in Functional Analysis. It also is the "best" Banach space, meaning that it has the most regular properties of all spaces.

7. $C(0,1)$ – the space of all continuous functions on the interval $[0,1]$, with the sup norm

$$\|x\|_\infty = \sup\{|x(t) : t \in [0,1]\},$$

is a Banach space.

8. The spaces $L_p(0,1)$, $1 \leq p < \infty$ of p–power Lebesgue integrable functions on $[0,1]$ with the norm

$$\|x\| = \left(\int_0^1 |x(t)|^p dt\right)^{\frac{1}{p}} .$$

Here we identify functions functions that differ only on a set of zero measure.

9. $L_\infty(0,1)$ – the space of all (essentially) bounded functions on $[0,1]$ with the norm

$$\|x\| = ess\sup\{|x(t)| : t \in [0,1]\} .$$

$L_\infty(0,1)$ is a Banach space.

The spaces listed above are only a few of the most typical Banach (or normed spaces) It should be strongly emphasized that there are many known examples of spaces which arise naturally in many areas of mathematics such as *e.g.* in Probability, Differential Equations and similar. One should mention here such natural generalizations of ℓ_p or L_p spaces as *Orlicz Spaces*. Besides these there are many Banach (and normed) spaces specifically constructed for special purposes (as to give an example or a counterexample). Some of these are quite involved and their construction is a result of serious efforts by many mathematicians. Usually such spaces are hard to describe in an introductory courses. Some names may be familiar to nonspecialists, such as *Tsirelson Space, Schlumprecht Space, the* X_1 *Space of Gowers and Maurey, etc.*

SEPARABLE SPACES.

All normed spaces are *topological vector spaces, i.e.* they have both an algebraic (vector) structure as well as a topological structure. These two structures are consistent (in the sense that the algebraic operations of addition of vectors and multiplication by scalars are continuous). This is an easy consequence of the norm axioms 2 and 3. The axiom 1 of the norm guarantees that the topology of a normed space is Hausdorff (T_2). Given these two parallel structures, one can investigate many properties of the spaces which can be expressed in algebraic terms (such as *convexity*) or in topological terms (*separability* or *compactness*), for example.This is , in fact, a very fruitful direction of research in Banach Space Theory. Here we will make only a few comments about the separability of Banach spaces.

Let us recall that a topological space is *separable* if there is a countable dense set in it. Or formally, X is separable if there is a subset $\{x_n\}_{n=1}^\infty$ in X such that

$\overline{\{x_n\}_{n=1}^{\infty}} = X$. In a topological vector space it simply means that here is a countable set such that its linear span, *i.e.* the set

$$\left\{ \sum_{i=1}^{n_i} \lambda_i x_i : n_i \in \mathbb{N}, \lambda_i \in \mathbb{R} \right\},$$

(which is called the *linear span* of $\{x_n\}$) is dense in X.

It is easy, by standard arguments of the elementary analysis courses, to show that of all the Banach spaces, presented in the previous paragraph, only ℓ_∞ and L_∞ spaces are not separable. This means, in particular, that there is an uncountable subset of the space such that any two elements of this subset are at least ϵ away, for every ϵ. In an ℓ_∞ space such a set is the set of all sequences consisting of zeros and ones. A distance between any two different elements of this type is one, and there are uncountably many of them.

We shall show here a nonstandard proof of the key Hölder inequality, mentioned earlier. While the usual proof is easy, the one given here is completely elementary and can be given to Calculus freshmen.

THEOREM *Let $p, q \in \mathbb{R}$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $x \in L_p$ and $y \in L_q$. Then $xy \in L_1$ and the norms of x and y satisfy*

$$\int |xy| \leq \left(\int |x|^p \right)^{\frac{1}{p}} \cdot \left(\int |y|^q \right)^{\frac{1}{q}}.$$

PROOF If either x or y has norm zero, then the inequality is trivially satisfied. We may therefore assume that both x and y have nonzero norms, and dividing by these nonzero norms we get that $\|x\| = \|y\| = 1$. Now

$$|xy| = e^{\ln |xy|} = e^{\ln |x| + \ln |y|} = e^{\frac{1}{p} \ln |x|^p + \frac{1}{q} \ln |y|^q} \leq \frac{1}{p} e^{\ln |x|^p} + \frac{1}{q} e^{\ln |y|^q} = \frac{1}{p}|x|^p + \frac{1}{q}|y|^q.$$

Notice that we only used the convexity of the exponential function. Hence we get

$$|xy| \leq \frac{1}{p}|x|^p + \frac{1}{q}|y|^q,$$

and now we integrate the above functions from 0 to 1 and we are done.

2. BOUNDED LINEAR OPERATORS.

DEFINITION Let X and Y be two normed spaces. A function $T : X \to Y$ is called a *linear operator* if

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y),$$

for all $x, y \in X$ and $\lambda, \mu \in \mathbb{R}$.

A linear operator is *continuous* if $x_n \to x \Rightarrow T(x_n) \to T(x)$ for every convergent sequence $\{x_n\}$ in X.

Clearly, due to linear nature of X, Y and T, a linear operator is continuous if and only if it is continuous at 0, *i.e.* $T(x_n) \to 0$ whenever $x_n \to 0$. Here $x_n \to 0$ means that $\|x_n\| \to 0$ and $T(x_n) \to 0$ means that $\|T(x_n)\| \to 0$. Note that we do not distinguish between the norms of X and Y although they are, in general, different. This is a widely accepted convention.

A linear operator $T : X \to Y$ is *bounded* if there exists a constant $M > 0$ such that $\|T(x)\| \leq M\|x\|$ for all $x \in X$. This implies several things. First that T is bounded on a *unit ball* of X, *i.e.* the set $B(0,1) = \{x \in X : \|x\| \leq 1\}$. That's why the operator is called bounded.

Secondly, it follows immediately that a bounded linear operator is continuous. It is less obvious but equally easy to show that a linear operator which is continuous is also bounded.

Hence we have

THEOREM *The Following Are Equivalent for a linear operator T between two normed spaces.*

1. *T is continuous ,*
2. *T is a continuous at a point $x \in X$,*
3. *T is continuous at 0 ,*
4. *T is bounded .*

The set of all bounded operators between two normed spaces is typically denoted by $\mathcal{L}(X, Y)$.

Let $T \in (X, Y)$. Then, since T is bounded, there exists a constant $M > 0$ such that $\|T(x)\| \leq M\|x\|$ for all $x \in X$. Set $\|T\| := \inf\{M \geq 0 : M\|x\| \geq \|T(x)\|\}$. (The infimum exists; the set of M's is bounded below by 0). The number $\|T\|$ associated with T is an *operator norm* (or simply a *norm*) of an operator T.

Therefore $(\mathcal{L}(X, Y), \|\cdot\|)$ is a normed space. If Y is a Banach space it is easy to show that $(\mathcal{L}(X, Y), \|\cdot\|)$ is a Banach space.

EXAMPLES OF LINEAR OPERATORS.

Let us recall that a linear operator between two finite–dimensional spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ can be represented as an $n \times m$ matrix. This is due to the fact that finite dimensional spaces posses *bases*. Such a representation (in the form of an *infinite* matrix) can, in many cases be obtained in general Banach spaces.

Much is known about linear operators between Banach Spaces. In fact, the main theorems of Functional Analysis : **The Closed Graph – Open Mapping Theorem** and **The Inverse Mapping Theorem** deal with the properties of linear operators. It should be mentioned, however, that not everything is yet known about the nature of operators on Banach spaces and some important questions are under extensive research.

BOUNDED LINEAR FUNCTIONALS.

DEFINITION A linear operator from a Banach space X to the field of real numbers $\mathbb{R}$ is called a *linear* functional.

Linear functionals are therefore very special cases of linear operators in Banach spaces. But because the are extremely important in the investigating the properties and structure of Banach spaces, as well as having powerful practical applications, they are investigated separately.

Functionals, like operators, can be defined on bases of vector spaces. But, because their range is 1–dimensional, they must take the value 0 infinitely often.

Consider the following

EXAMPLE Let $X = \mathbb{R}^2$ and let us define a linear functional on X as follows :

$$f((x,y)) = \begin{cases} 5 & \text{if } x = 1, y = 0 \\ -3 & \text{if } x = 0, y = 1 \end{cases}$$

Then, *e.g.* $f((2,3)) = 2f((1,0)) + 3f((0,1)) = 2 \cdot 5 + 3 \cdot (-3) = 1$. Also, $f((x,y)) = xf((1,0)) + yf((0,1))$ and this implies that $0 = 5x - 3y$. But this is the equation of a line in $\mathbb{R}^2$ passing through the origin, and so we have showed that a linear functional *vanishes* on a subspace of *co–dimension* 1 in $\mathbb{R}^2$. This is, as we shall show later, a general property of all linear functionals in Banach spaces : their *kernel* defined as $ker(f) = \{x \in X : f(x) = 0\}$ has a co-dimension one. Every linear functional f on a normed space X forces decomposition of X as follows

$$X = \{x_0\} + ker(f) \,,$$

where $\{x_0\}$ is a subspace of X generated by a point x_0 in X for which $f(x_0) \neq 0$. Notice that the above is the algebraic direct sum of $\{x_0\}$ and $ker(f)$. If X is a Banach space and $ker(f)$ is closed then X can be similarly represented as a *topological direct sum* of $\{x_0\}$ and $ker(f)$.

There is an important characterization of the continuity of linear functionals on a Banach space

THEOREM *A functional f on a Banach space X is continuous if and only if its kernel, $ker(f)$, is a closed subspace of X.*

But not all linear functionals in a Banach space are continuous. In fact, we have the following

EXAMPLE In every normed infinite–dimensional (in particular, Banach) space X there is a discontinuous linear functional.

PROOF Let $H = \{x_\alpha : \alpha \in A\}$ be a Hamel basis of X. Then there is a subset $\{x_{\alpha_n} : n \in \mathbb{N}\}$ of H and positive scalars $\{\lambda_n : n \in \mathbb{N}\}$ with the property that the set $\{\lambda_n x_{\alpha_n} : n \in \mathbb{N}\}$ is linearly independent and

$$\|\lambda_n x_{\alpha_n}\| < \frac{1}{n} \quad (n = 1, 2, ...).$$

Set $f(\lambda x_{\alpha_n}) = 1$ for $n = 1, 2, ...$ and extend f algebraically to the whole space X. Then $f \in X^*$ but f is not continuous.

DUAL SPACE, HAHN – BANACH THEOREM .

DEFINITION. Let X be a Banach space. The set of all linear functionals $f : X \to \mathbb{R}$ is denoted by X^* and is called the *algebraic dual space* (or simply, the *algebraic dual*) of X. The set of all continuous linear functionals on X is referred to a the *dual space* of X or, simply, *dual* of X and is denoted X'.

Every Banach space has a continuous linear functional, in fact, it has very many. Continuous linear functionals in some sense perform the role of real numbers, they allow to "measure" objects in a Banach space. Therefore it is important to know that there are linear functionals in the space, and what form they have. The answer to the first question is provided by

THEOREM (HAHN – BANACH) *Let X be a Banach space and X_0 its closed subspace. Let f_0 be a continuous linear functional on X_0 with the norm $\|f_0\|$. Then there exists a continuous linear functional f on X such that f restricted to X_0 is equal to f_0. Moreover, the norm of f is equal to the norm of f_0, i.e. $\|f_0\| = \|f\|$.*

The Hahn–Banach Theorem is one of the most important theorems in Functional

Analysis and has been intensely investigated since it has been first discovered in the early 1920's.

It ia easy to construct continuous linear functionals in finite dimensional spaces. If we have one, the Hahn–Banach theorem provides an extension of the functional to the whole space. In other words, every Banach space has plenty of continuous linear functionals.

While we know that there are many continuous linear functionals on a given Banach space, we would like to know their form, if possible. In the terminology of Functional Analysis we would like to know the dual of a given Banach space. It is, in fact, an easy problem, at least for most of the spaces listed in CHAPTER 1.

DUALS OF SOME BANACH SPACES

1. $c_0' = c' = \ell_1' = \ell_\infty$
2. $\ell_p' = \ell_q$, where $\frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty$,
3. $L_p' = L_q$, where $\frac{1}{p} + \frac{1}{q} = 1$,
4. $L_1' = L_\infty$.

5. THE WEAK AND WEAK* TOPOLOGIES.

Besides the usual topology (convergence) in the Banach space, *i.e.* the one determined by the norm, it is convenient to consider several other topologies of the vector space $(X, \|\cdot\|)$. Let us recall that a sequence $\{x_n\}$ of elements of a normed space $(X, \|\cdot\|)$ converges (to an element x_0) in X if the real sequence $\|x_n - x_0\|$ converges to 0. This is precisely the concept of the *norm convergence*. If a functional f is continuous on X it follows from the norm convergence of x_n that $f(x_n)$ converges to $f(x_0)$. This is the basis of the concept of *weak convergence.*

DEFINITION A sequence x_n of elements of a normed space $(X, \|\cdot\|)$ converges *weakly* to x_0 if $f(x_n)$ converges to $f(x_0)$ for every $f \in X'$.

Notice that this concept has a specific topological meaning. The weak convergence, as defined above is implied by the norm convergence, but not *vice versa.* This means that a sequence can be weakly convergent but not norm convergent. In fact, it usually is.

THEOREM (SCHUR) *In the space ℓ_1 the norm and weak convergence coincide for sequences.*

Therefore, we have

DEFINITION A Banach space X is called a *Schur* space if the norm and weak convergence of sequences coincide in X.

It should be pointed out that Schur spaces are rare. Only a few of the known spaces have this property.

The weak convergence is a very useful concept that has been intensely investigated and much is known about it. But it is perhaps better to consider it in the setting of *vector topology.*

DEFINITION A topology on a normed space X is called a *weak topology* if it is the weakest topology for which all norm continuous functionals on X remain continuous. The weak topology is usually denoted by $\sigma(X, X')$.

It is a well known fact in General Topology that such weak topology always exists. There are several useful properties of the weak topology that we list here for the sake of completeness.

THEOREM *A subset of a normed space is bounded if and only if it is weakly bounded .*

THEOREM *A linear operator T from a normed space X into a normed space Y is norm–to–norm continuous if and only if it is weak–to–weak continuous .*

The weak topology of a normed space is Hausdorff. This is easy to see. If x_1 and x_2 are two different elements of the normed space X, the Hahn – Banach theorem guarantees that there is continuous linear functional $f \in X'$ such that $f(x_1) \neq f(x_2)$. This is, of course, another way of saying that two points in X can be separated by open neighbourhoods. But a weak topology is, in general neither complete, nor normable *i.e.*, $(X, \sigma(X, X'))$ is not a Banach space. Indeed we have

THEOREM *The weak topology of a normed space is complete if and only if the space is finite–dimensional.*

As a further consequence we have that

THEOREM *The weak topology of a normed space is normable if and only if the space is finite–dimensional.*

So only in the space $\mathbb{R}^n$ of all the spaces listed above is the weak topology normable. It is, in fact, identical to the original (Euclidean) topology.

THE SECOND DUAL X".

Let X be a Banach space with a dual X'. We recall that the space of all linear operators $\mathcal{L}(X,Y)$ between two Banach spaces X and Y is again a Banach space. Here we have $Y = \mathbb{R}$, and $\mathbb{R}$ is a Banach space (with the Euclidean topology). Hence the space $(X', \|\cdot\|)$ is a Banach space. We can consider all continuous linear functionals on $(X', \|\cdot\|)$. This is a dual space to X' and, quite properly, we call it the *Second Dual* of X and denote X''. Formally

DEFINITION . Let X be a normed space. The *second dual* or a *bidual* of X is the dual space $(X')'$.

For any normed space X its second dual is a Banach space. This means, of course, that there is a natural norm on this space, under which it is complete.

Let us now slightly change the notation and denote the continuous linear functionals on X by x' rather than f. This is to emphasize that they are members of X'. Consider :

$$(\star) \qquad x''(x') := x'(x), \quad x \in X, x' \in X'.$$

This way we have defined a continuous linear functional x'' on X', using the value of x' on x in X. It is, therefore, a well defined function.

DEFINITION The map $x \mapsto x''$ from X to X'' is called the *natural map* or *canonical embedding* from X to X''.

Since we consider x'' a linear functional on X' defined by $(\star)$, this leads to another concept of weak convergence, this time on X'.

DEFINITION The topology for X' which is the weakest topology on X' such that, for each $x \in X$ the linear functional $x' \mapsto x'(x)$ on X' is continuous with respect to this topology is called the *weak** topology of X' (or the topology $\sigma(X', X)$). This is pronounced "weak star".

Similar to the weak topology. the weak* topology has a number of useful properties of which we will list a few.

THEOREM *Let X be a normed space. Them the weak* and norm topologies of X' coincide if and only if the space X is finitely–dimensional .*

THEOREM *Let X be a Banach space. Then a subset of X' is bounded if and only if it is weakly* bounded.*

THEOREM *Let X be a Banach space. Then $(X', \sigma(X', X))$ is a Banach space if and only if X is finite–dimensional .*

It should be strongly emphasized, that despite certain similarities, the weak and weak* topologies are different topologies; the first one is on X and the other one on X'.

SCHAUDER BASES IN BANACH SPACES .

Let X be a Banach space. It has, as a vector space, a basis, *i.e.* maximal linearly independent set. Such a basis is referred to as a *Hamel basis* of X. It is an algebraic basis but it should be pointed out that a complete normed space of infinite dimension is necessarily of uncountable dimension. An incomplete normed space may be of countable dimension.

Besides algebraic bases, one considers another type of the basis in a Banach space.

DEFINITION Let X be a Banach space. A sequence $\{e_n\}$ of elements of X is called a *Schauder basis* (or, simply, a *basis*) if every element of X can be uniquely represented as a series $\sum_{n=1}^{\infty} a_n e_n$, where $a_n \in \mathbb{R}, n = 1, 2, \ldots$.

Not every Banach space has a basis. It is easy to see from the definition that if a space X has a basis it must be separable. It is so because $\overline{lin}\{x_n\} = X$ and $\{x_n\}$ is, of course, countable.

But most separable Banach spaces have bases. In fact, the spaces c_0, ℓ_p , where $1 \leq p < \infty$ have the simplest basis : the set of all unit vectors $\{\delta_{i,j}\}, i, j = 1, 2, \ldots$. A more complicated bases also exist in $L_p(0,1), 1 \leq p < \infty$: the *Haar system*. The space $C(0,1)$ also has a basis. The spaces ℓ_∞ and $L_\infty(0,1))$ are nonseparable, and as such, do not have a basis. But it was for a long time an open problem whether every separable Banach space has a basis. It was only in 1973 that P. Enflo showed that there exists a separable Banach space without a basis. Today we know more spaces of this kind, some of them quite natural. But it was already known to Banach that every Banach space (separable or not) has a weaker property. Let us define it.

DEFINITION A Banach space X has a *basic sequence* if there exists a sequence $\{e_n\}$ of elements of X such that $\{e_n\}$ is a basis for its closed linear span.

THEOREM (BANACH) *Every infinite – dimensional Banach space contains a basic sequence.*

The proof of the above theorem is essentially a simple argument about existence of continuous linear functionals in a space. To see that linear continuous functionals and bases are closely linked let us review the fundamental characterization of basic sequences due to S. Mazur.

THEOREM (MAZUR) *A sequence $\{e_n\}$ in a Banach space X is basic if and only if each e_n is nonzero and there is a real number M such that*

$$\|\sum_{n=1}^{m_1} a_n e_n\| \leq M \|\sum_{n=1}^{m_2} a_n e_n\|$$

whenever $m_1, m_2 \in \mathbb{N}, m_1 \leq m_2$ and $a_1, a_2, \cdots \in \mathbb{R}$.

If you let $m_2 \to \infty$ in the above inequality, it shows that if e_n is a basic sequence then for every $m \in \mathbb{N}$ we must have

$$\|\sum_{n=1}^{m} a_n e_n\| \leq M\|x\|\,,$$

and this simply means that he operator $P_m : X \to X$ defined by

$$P_m(x) = \sum_{n=1}^{m} a_n e_n$$

is a continuous linear operator with the norm $\|P_m\| \leq M$.

This operator is called a *projection* onto a subspace of dimension m. It also produces another continuous projection onto the complementary subspace of codimension n, *i.e.* $I_X - P_m$. Thus the existence of a basic sequence guarantees the existence of some continuous linear operators on the space, the projections on spaces of finite dimension and finite codimension. If we want to have more interesting operators on a space then the basic sequences are not sufficient.

Many bases, such as standard unit sequence bases of c_0 or $\ell_p, 1 < p < \infty$ have much stronger properties. One of them is that these Schauder bases are *unconditional*. It is an elementary fact from Calculus that a series which converges need not converge absolutely, *i.e.* the series of absolute values may diverge. A good example is an alternating harmonic series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$. It converges but the series of absolute values $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. A classical theorem of Dirichlet says that if a series in $\mathbb{R}$ converges conditionally, it can be rearranged so that it converges to any limit, or diverges. In an infinite dimensional space, the situation is more complicated. A series may converge, (*i.e.* the sequence $\|\sum_{n=1}^{m} x_n\|$ converges to an element s as $m \to \infty$), it may converge absolutely (*i.e.* the series $\sum_{n=1}^{\infty} \|x_n\|$ converges), or it may converge *unconditionally*, *i.e.* the series $\sum_n x_{\pi(n)}$ converges for any *permutation* $\pi : \mathbb{N} \to \mathbb{N}$.

This leads to the following

DEFINITION A basic sequence $\{e_n\}$ in X is an *unconditional* basic sequence if the series
$$\sum_n a_{\pi(n)} e_{\pi(n)}$$
converges for any permutation $\pi : \mathbb{N} \to \mathbb{N}$ and any sequence a_n of real numbers .

Unconditional basic sequences are characterized by the following theorem, due to Stanisław Mazur.

THEOREM (MAZUR) *The Following Are Equivalent for a sequence $\{e_n\}$ in a Banach Space*

1. *$\{e_n\}$ is unconditional basic sequence ,*
2. *$\sum_{n=1}^{\infty} \epsilon_n a_n e_n$ converges for any $|\epsilon_n| = 1$,*
3. *$\sum_n a_{\pi(n)} e_{\pi(n)}$ converges for any permutation π of $\mathbb{N}$,*
4. *$\sum_{n\in A} a_n e_n$ converges for any subset A of $\mathbb{N}$.*

The property $\{4\}$ of unconditional basic sequences has powerful consequences. It simply means that the following holds
$$\|\sum_{n\in A} a_n e_n\| \leq M \|\sum_{n=1}^{\infty} a_n e_n\| = M\|x\|$$
for some $M > 0$ and every $m \in \mathbb{N}$ and every subset $A \subset \mathbb{N}, x \in X$.

This means that we again have continuous linear projections $P_A : X \to X$ but this time the projections are on subspaces of infinite dimensional subspaces, *i.e.* they have infinite *rank* and *corank*. Therefore a space with an unconditional basis will have many "interesting" operators on it.

For a long time it was unknown whether any Banach space must have an unconditional basic sequence. Finally, in 1993, W.T. Gowers and B. Maurey proved the following

THEOREM (GOWERS – MAUREY) *There exists a separable Banach space without any unconditional basic sequence .*

7. UNIFORM CONVEXITY AND REFLEXIVITY.

DEFINITION A linear operator T between two Banach spaces X and Y is an *isomorphism* or *normed space isomorphism* if it one–to–one and continuous and its inverse T^{-1} is continuous on the range of T. The operator T is an *isometric isomorphism* or *linear isometry* if $\|T(x)\| = \|x\|$ whenever $x \in X$. The spaces X and Y are *isomorphic* if there is an isomorphism from X onto Y, and *isometrically isomorphic* if there is an isometric isomorphism from X onto Y.

Let us recall the natural map from X into X'' defined on p. 10. If this map is onto then we have a special kind of space.

DEFINITION A normed space X is *reflexive* if the natural map from X into X'' is onto X''.

It is easy to observe that since X'' is a Banach space then a reflexive normed space must be a Banach space.

It is also easy to see (cf. list of duals on p.8) that the spaces $\ell_p, 1 < p < \infty$ and $L_p(0,1), 1 < p < \infty$ are reflexive. The remaining spaces, including ℓ_1 and $L_1(0,1)$ are not reflexive.

Note also that in the definition of reflexivity we requested that the spaces X and X'' be isomorphic under the natural map. A space may be isomorphic to its second dual by some isomorphism and not be reflexive. The first example of this sort was shown by R.C. James in 1950.

However, reflexive spaces are very regular. We shall mention, without proofs, several properties of reflexive spaces.

THEOREM *Reflexivity is an invariant of isomorphisms, i.e. every normed space isomorphic to a reflexive space is itself reflexive.*

THEOREM (PETTIS) *Every closed subspace of a reflexive normed space is reflexive.*

COROLLARY *If X is a Banach space, then X is reflexive if and only if X' is reflexive .*

This last Corollary says that that there are only two types of Banach spaces : such that only X and X' are different (because $X \cong X''$ and $X' \cong X'''$ etc.), and such that every space in the sequence $X, X', X'', X''', \ldots$ is different (nonisomorphic).

A unit ball in a Banach space need not be a "ball" at all. In fact, a unit ball in $\mathbb{R}^2$ equipped with a supremum norm is a square. In order to capture the ball–like characteristic of the Euclidean ball we need an additional concept.

DEFINITION A normed space X is *strictly convex* or *rotund* if $\|\alpha x_1+(1-\alpha)x_2\| < 1$ whenever x_1 and x_2 are different points of the unit ball $B(0,1)$ of X and $0 < \alpha < 1$.

DEFINITION Let X be a normed space. Define a function $\delta_X : [0,2] \to [0,1]$ by the formula

$$\delta_X(\epsilon) = \inf\left\{1 - \|\frac{1}{2}(x+y)\| : x, y \in B(0,1), \|x-y\| \geq \epsilon\right\}$$

if $X \neq \{0\}$, and by the formula

$$\delta_X(\epsilon) = \begin{cases} 0 & \text{if } \epsilon = 0, \\ 1 & \text{if } 0 < \epsilon \leq 2, \end{cases}$$

if $X = \{0\}$. Then δ_X is the *modulus of convexity* or *modulus of rotundity* of X. The space X is *uniformly convex* or *uniformly rotund* if $\delta_X(\epsilon) > 0$ whenever $0 < \epsilon \leq 2$.