Semigroup Crossed products

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Introduction

In the theory of C^*-dynamical systems and their crossed products, we have some different kinds of semigroup crossed products. My main focus is on crossed products by the **positive cones** of totally ordered abelian discrete groups.

Recall that if Γ is a totally ordered (abelian discrete) group, then its **positive cone** is the subset $\{s \in \Gamma : s \geq e\}$ denoted by Γ^+, which is a semigroup (subsemigroup of Γ).

For example, the positive cone of the group $\mathbb{Z}$ is $\mathbb{Z}^+ = \{0, 1, 2, ...\}$, for which the notation $\mathbb{N}$ is usually used in our work.

Introduction

Associated to such semigroups, there are two kinds of semigroups:

1. the isometric crossed product;
2. the partial-isometric crossed product.

In this talk, as the isometric crossed product was introduced earlier than the partial-isometric crossed product, we give an introduction on the isometric crossed product, but only the isometric crossed product of a unital C^*-algebra by the semigroup $\mathbb{N} = \mathbb{Z}^+$.

C^*-dynamical systems

Let A be a unital C^*-algebra, and α an (injective) endomorphism of A, which is not necessarily unital (we may not have $\alpha(1) = 1$). Then α generates an action of $\mathbb{N}$ on A by endomorphisms such that

$$\alpha_0 := \mathrm{id}_A \ \text{ and; } \ \alpha_n := \alpha \circ \alpha \circ ... \circ \alpha \ (n \geq 1 \text{ times}).$$

So for every $n \in \mathbb{N}$ and $a \in A$, we have $n \cdot a := \alpha_n(a)$. Therefore we get the C^*-dynamical system $(A, \mathbb{N}, \alpha)$, which sometimes could be denoted simply by (A, α).

Example 1. Let $\mathbf{c}$ be the C^*-algebra of all convergent sequences of complex numbers. Then the forward shift $\tau : \mathbf{c} \to \mathbf{c}$

$$\tau(x_0, x_1, ...) = (0, x_0, x_1, ...),$$

gives an action of $\mathbb{N}$ on $\mathbf{c}$ by endomorphisms. Also note that $\tau(1, 1, ...) = (0, 1, 1, ...)$ which implies that τ is not unital.

We represent a C^*-dynamical system $(A, \mathbb{N}, \alpha)$ by operators on a Hilbert space in a similar fashion to the one for C^*-dynamical system of groups actions, in which the action is implemented by isometries rather than unitaries.

Let us first recall that a bounded operator V on a Hilbert space H is called an *isometry* if $\|V(h)\| = \|h\|$ for all $h \in H$, which is equivalent to $V^*V = 1$. Also the product (composition) of two isometries is always an isometry.

Example 2. The truncated shift T on $\ell^2(\mathbb{N}) := \{(\lambda_n)_{n\in\mathbb{N}} : \sum_{n=0}^{\infty} |\lambda_n|^2 < \infty\}$ is an isometry:

$$T(\lambda_0, \lambda_1, \lambda_2, ...) = (0, \lambda_0, \lambda_1, \lambda_2, ...)$$

Covariant representations

An *isometric representation* of $\mathbb{N}$ on a Hilbert space H is a map $V : \mathbb{N} \to B(H)$ such that $V_n := V(n)$ is an isometry, and $V_{m+n} = V_m V_n$ for all $m, n \in \mathbb{N}$.

And now:

Definition

Let $(A, \mathbb{N}, \alpha)$ be a C^*-dynamical system. A *covariant isometric representation* of $(A, \mathbb{N}, \alpha)$ on a Hilbert space H is a pair (π, V) consisting of a nondegenerate (unital) representation $\pi : A \to B(H)$ and an isometric representation $V : \mathbb{N} \to B(H)$, such that

$$\pi(\alpha_n(a)) = V_n \pi(a) V_n^* \quad \text{for all } a \in A, n \in \mathbb{N}.$$

Example 3. If

$$\pi : \mathbf{c} \to B(\ell^2(\mathbb{N}))$$

$$\pi(x_0, x_1, ...)(\lambda_0, \lambda_1, \lambda_2, ...) = (x_0\lambda_0, x_1\lambda_1, x_2\lambda_2, ...); \quad \text{and}$$

$$T : \mathbb{N} \to B(\ell^2(\mathbb{N}))$$

$$T_1(\lambda_0, \lambda_1, \lambda_2, ...) = (0, \lambda_0, \lambda_1, \lambda_2, ...)$$

The pair (π, T) is a covariant isometric representation of the system $(\mathbf{c}, \mathbb{N}, \tau)$ on $\ell^2(\mathbb{N})$.

Covariant representations

A crucial point: As isometric crossed product (similar to crossed products by groups) is defined as a universal object for covariant isometric representations of the system, this is natural to ask whether any system $(A, \mathbb{N}, \alpha)$ admits a nontrivial covariant isometric representation (π, V), meaning that with $\pi \neq 0$.

The answer is that there are some systems which do not have a nontrivial covariant isometric representation. Consequently for such systems, the associated isometric crossed product is the trivial algebra (zero algebra), and therefore there is no point to define the isometric crossed product of the system.

For example, the system $(\mathbf{c}_0, \mathbb{N}, \sigma)$, in which the action σ is given by backward shift, **does not** have any nontrivial covariant isometric representation:

$$\sigma(x_0, x_1, x_2, ...) = (x_1, x_2, ...)$$

The isometric crossed product

Definition

The isometric crossed product for a dynamical system $(A, \mathbb{N}, \alpha)$ is a C^*-algebra B together with a faithful nondegenerate (unital) $*$-homomorphism $i_A : A \to B$ and an isometry valued homomorphism $i_{\mathbb{N}} : \mathbb{N} \to B$ satisfying

1. $(i_A, i_{\mathbb{N}})$ is covariant into B: $i_A(\alpha_n(a)) = i_{\mathbb{N}}(n) i_A(a) i_{\mathbb{N}}(n)^*$;
2. for every covariant representation (π, V) of $(A, \mathbb{N}, \alpha)$ on H there is a nondegenerate (unital) representation $\pi \times V$ of B on H such that
$$(\pi \times V) \circ i_A = \pi \ \text{ and } \ (\pi \times V) \circ i_{\mathbb{N}} = V;$$
3. $B = \overline{\text{span}}\{i_{\mathbb{N}}(n)^* i_A(a) i_{\mathbb{N}}(m) : a \in A, m, n \in \mathbb{N}\}$.

The isometric crossed product

If the isometric crossed product of a system $(A, \mathbb{N}, \alpha)$ exists (nontrivial algebra), then it is unique up to isomorphism:

If C is another C^*-algebra together with a pair $(j_A, j_{\mathbb{N}})$ of $(A, \mathbb{N}, \alpha)$ into C satisfying (1), (2), and (3), then there is an isomorphism $\varphi : C \to B$ such that

$$\varphi \circ j_A = i_A \ \text{ and } \ \varphi \circ j_{\mathbb{N}} = i_{\mathbb{N}}.$$

We therefore write $A \times_{\alpha}^{\text{iso}} \mathbb{N}$ for the isometric crossed product B to differentiate it with other types of semigroup crossed products.

Some examples of isometric crossed products

Example 4. The isometric crossed product $\mathbf{c} \times_{\tau}^{\text{iso}} \mathbb{N}$ of the system $(\mathbf{c}, \mathbb{N}, \tau)$ is isomorphic to the Toeplitz algebra $\mathcal{T}(\mathbb{Z})$, which is the universal C^*-algebra generated by an isometry.

Example 5(nonunital). Let $\mathbf{c}_0$ be the C^*-algebra of all sequences convergent to 0. Then forward shift τ gives an action of $\mathbb{N}$ by endomorphisms on $\mathbf{c}_0$ (In fact, $\mathbf{c}_0$ is an invariant τ-extendible ideal of $\mathbf{c}$). So for the isometric crossed product of the system $(\mathbf{c}_0, \mathbb{N}, \tau)$, we have

$$\mathbf{c}_0 \times_{\tau}^{\text{iso}} \mathbb{N} \simeq \mathcal{K}(\ell^2(\mathbb{N})) \text{ (compact operators).}$$

Example 6. If α is an automorphism of a C^*-algebra A, then the isometric crossed product $A \times_{\alpha}^{\text{iso}} \mathbb{N}$ of the system $(A, \mathbb{N}, \alpha)$ is isomorphic to the group crossed product $A \times_{\alpha} \mathbb{Z}$.

Thank you