"Strange structures" in 2– and 3–spaces or how to refine box–meshes

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Outline

Intro

- a motivating question

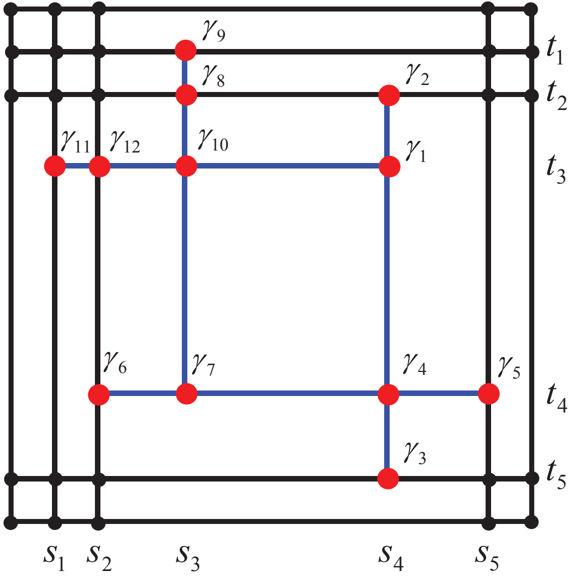
Main

- a story
- a problem
- some results

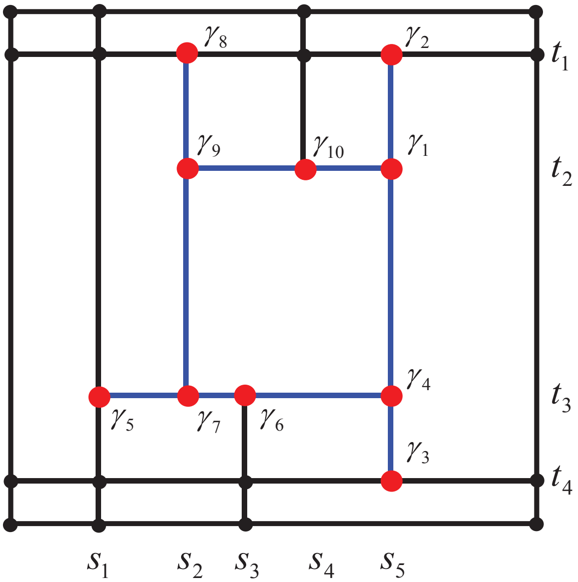
Final

- discussion of the motivating question

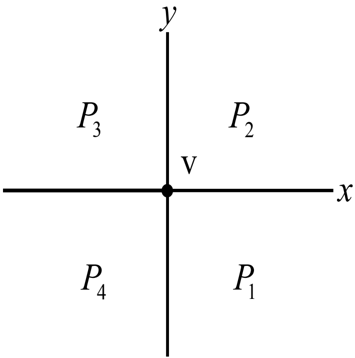
An unstable T–mesh for $m = m' = 5$ and $r = r' = 3$:



An unstable T–mesh for $m = m' = 4$ and $r = r' = 2$:



preliminaries



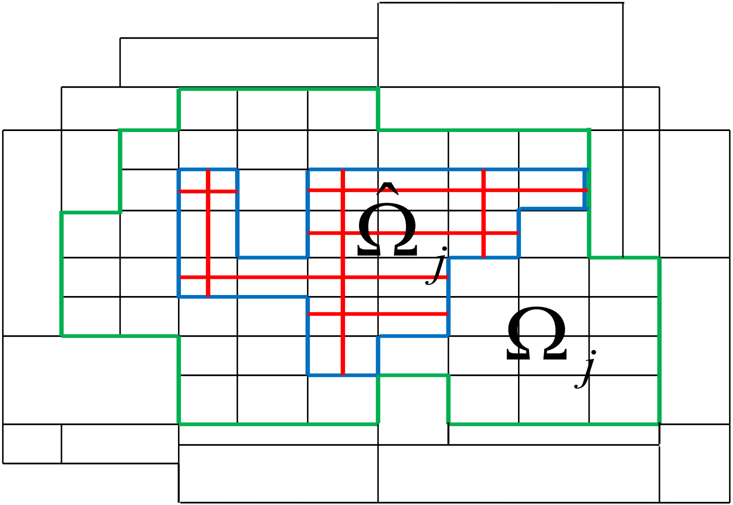
$$k_v = \frac{1}{(m!)^2} \frac{\partial^{2m}(P_2(x,y) + P_4(x,y) - P_1(x,y) - P_3(x,y))}{\partial x^m \partial y^m} \Big|_{x=x_0, y=y_0}$$

refinement strategy

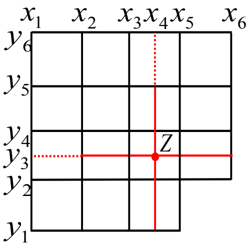
Let m be an integer $\geqslant 2$ (for example $m = 2$ and $m = 3$).

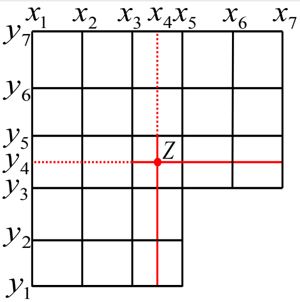
1. The initial T-mesh T_0 occupies the domain Ω_0, which is composed of the cells of a tensor–product mesh and admits an offset at a distance $\frac{m-1}{2}$.
2. Induction from any $j \geqslant 0$ to $j+1$ proceeds as follows. For a given T-mesh T_j, it is necessary to find domains $\widehat{\Omega}_j \subset \Omega_j$ which satisfy the following assumptions: domains $\widehat{\Omega}_j$ and Ω_j are both constructed from the cells of some tensor–product mesh, such that each cell in Ω_j is also a cell of the T-mesh T_j; and Ω_j admits an inner offset at a distance $\frac{m-1}{2}$. The refined T-mesh T_{j+1} is derived from T_j by dividing each cell of the domain $\widehat{\Omega}_j$ such that the line–segments $E_{j1}, \ldots, E_{jn_j}$ refining T_j have endpoints on $\partial\widehat{\Omega}_j$. It is supposed that the line–segments $E_{j1}, \ldots, E_{jn_j}$ are maximal interior segments of the T-mesh T_{j+1}, i.e. $E_{j1}, \ldots, E_{jn_j}$ do not intersect with the boundary $\partial\Omega_0$.

refinement strategy



refinement strategy





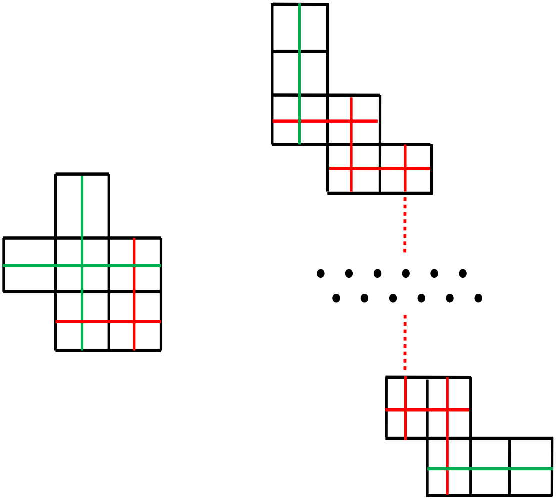
left:

$$f_Z(x,y) = N[x_1,x_2,x_3,x_4,x_5](x)N[y_2,y_3,y_4,y_5,y_6](y) + \frac{3}{5}N[x_2,x_3,x_4,x_5,x_6](x)$$
$$N[y_2,y_3,y_4,y_5,y_6](y) + \frac{3}{5}N[x_1,x_2,x_3,x_4,x_5](x)N[y_1,y_2,y_3,y_4,y_5](y)$$

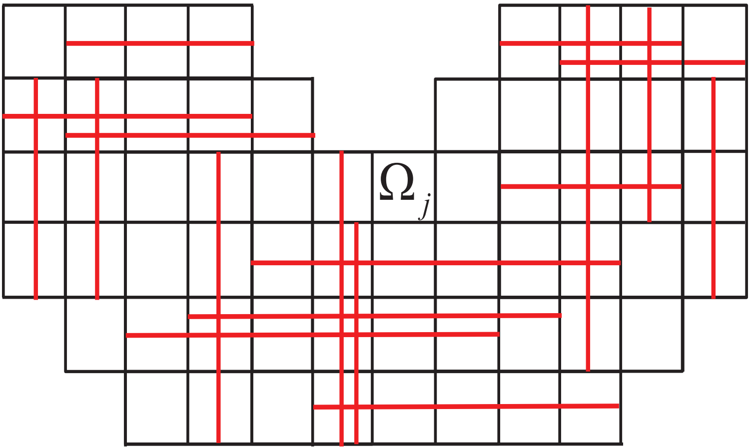
right:

$$f_Z(x,y) = -N[x_1,x_2,x_3,x_4,x_5](x)N[y_3,y_4,y_5,y_6,y_7](y) + N[x_3,x_4,x_5,x_6,x_7](x)$$
$$N[y_3,y_4,y_5,y_6,y_7](y) + N[x_1,x_2,x_3,x_4,x_5](x)N[y_1,y_2,y_3,y_4,y_5](y)$$

refinement strategy



refinement strategy



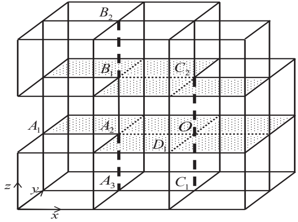
preliminaries

$$
\begin{array}{ccccccccccc}
 & & & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{I}_m(\mathcal{T}^o): & \mathbf{0} & \xrightarrow{\widehat{\partial}_3} & \bigoplus_{\sigma\in\mathcal{T}_2^o}[\sigma]\,\mathcal{J}_m(\sigma) & \xrightarrow{\widehat{\partial}_2} & \bigoplus_{\tau\in\mathcal{T}_1^o}[\tau]\,\mathcal{J}_m(\tau) & \xrightarrow{\widehat{\partial}_1} & \bigoplus_{\gamma\in\mathcal{T}_0^o}[\gamma]\,\mathcal{J}_m(\gamma) & \xrightarrow{\widehat{\partial}_0} & \mathbf{0} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{R}_m(\mathcal{T}^o): & \bigoplus_{\mu\in\mathcal{T}_3}[\mu]\,\mathcal{R}_m & \xrightarrow{\partial_3} & \bigoplus_{\sigma\in\mathcal{T}_2^o}[\sigma]\,\mathcal{R}_m & \xrightarrow{\partial_2} & \bigoplus_{\tau\in\mathcal{T}_1^o}[\tau]\,\mathcal{R}_m & \xrightarrow{\partial_1} & \bigoplus_{\gamma\in\mathcal{T}_0^o}[\gamma]\,\mathcal{R}_m & \xrightarrow{\partial_0} & \mathbf{0} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{S}_m(\mathcal{T}^o): & \bigoplus_{\mu\in\mathcal{T}_3}[\mu]\,\mathcal{R}_m & \xrightarrow{\overline{\partial}_3} & \bigoplus_{\sigma\in\mathcal{T}_2^o}[\sigma]\,\mathcal{R}_m/\mathcal{J}_m(\sigma) & \xrightarrow{\overline{\partial}_2} & \bigoplus_{\tau\in\mathcal{T}_1^o}[\tau]\,\mathcal{R}_m/\mathcal{J}_m(\tau) & \xrightarrow{\overline{\partial}_1} & \bigoplus_{\gamma\in\mathcal{T}_0^o}[\gamma]\,\mathcal{R}_m/\mathcal{J}_m(\gamma) & \xrightarrow{\overline{\partial}_0} & \mathbf{0} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & &
\end{array}
$$

$$H_3(\mathfrak{S}_m(\mathcal{T}^0)) = \ker\overline{\partial}_3 = S_m(\mathcal{T})$$

$$\dim \mathcal{S}_m(\mathcal{T}) = (m+1)^3 f_3 - m(m+1)^2 f_2^0 + m^2(m+1) f_1^0 - m^3 f_0^0 + \dim(H_2(\mathfrak{S}_m(\mathcal{T}^0))) - \dim(H_1(\mathfrak{S}_m(\mathcal{T}^0))) + \dim(H_0(\mathfrak{S}_m(\mathcal{T}^0))).$$

dimension



Theorem

For a given domain Ω, suppose that $\partial\Omega$ is a two-dimensional sphere (topologically). In addition, suppose that for any plane S' splitting Ω the corresponding two-dimensional domain S is simply connected. Then,

$$\dim \mathcal{S}_m(\mathcal{T}) = (m+1)^3 f_3 - m(m+1)^2 f_2^0 + m^2(m+1) f_1^0 - m^3 f_0^0,$$

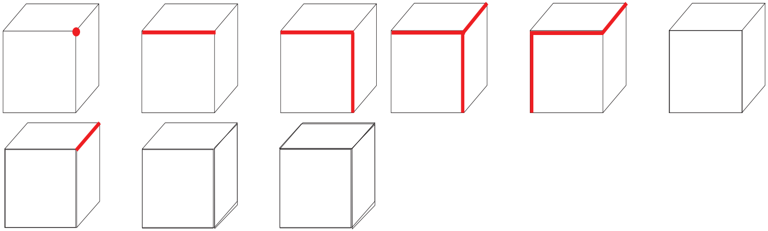
where f_3, f_2^0, f_1^0, and f_0^0 are the numbers of cells, inner facets, inner edges, and inner vertices of a domain Ω, respectively.

Theorem

For a given $m \geqslant 1$, suppose that $\Omega \in \mathcal{A}^3_{m-1}$, and let $\mathcal{T}$ be the corresponding three-dimensional mesh. Then, the dimension of a spline space $\mathcal{S}_m(\mathcal{T})$ is

$$\dim \mathcal{S}_m(\mathcal{T}) = (m+1)^3 f_3 - m(m+1)^2 f_2^0 + m^2(m+1) f_1^0 - m^3 f_0^0,$$

where f_3, f_2^0, f_1^0, and f_0^0 are the numbers of cells, inner facets, inner edges, and inner vertices of a domain Ω, respectively.



basis

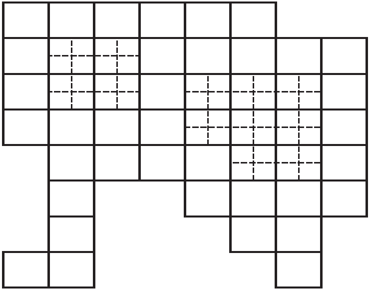
Theorem

Let Ω be a three-dimensional domain that admits an offset at a distance of $\frac{m-1}{2}$; namely, $\Omega \in \mathcal{A}^3_{m-1}$. Then, the following identity holds: $\mathcal{N} = (m+1)^3 f_3 - m(m+1)^2 f_2^0 + m^2(m+1) f_1^0 - m^3 f_0^0$.

Corollary

Suppose that $\Omega \in \mathcal{A}^3_{m-1}$. Then, the basis of a space $\mathcal{S}_m(\mathcal{T})$ can be obtained as follows:
$\{b|_\Omega : b(x,y,z) \in \mathcal{B} \ \wedge \ \operatorname{supp} b(x,y,z) \cap \operatorname{int} \Omega \neq \varnothing\}$.

Kraft's selection mechanism



Theorem

Let $m \geqslant 1$ be an integer. For a given nested sequence of domains $\Omega^0 \supset \Omega^1 \supset \cdots \supset \Omega^{N-1} \supset \Omega^N = \varnothing$ suppose that the domain $R^\ell = \Omega^0 \backslash \Omega^{\ell+1}$ (with respect to the grid G^ℓ) admits an offset at a distance of $\frac{m-1}{2}$ for each $\ell = 0, \dots, N-1$. Then, the set of B-splines from $\mathcal{K}$ restricted on Ω^0 is a basis of the spline space W defined over the corresponding hierarchical mesh $\mathcal{H}$. The theorem holds for $d = 1, 2, 3$, if that is the case.

Theorem

Let $\Omega \in \mathcal{A}^2_{m-1,n-1}$ be a two-dimensional domain and $\mathcal{T}$ be the corresponding T-mesh. Then, the dimension of a space $\mathcal{S}_{m,n}(\mathcal{T})$ is

$$\dim \mathcal{S}_{m,n}(\mathcal{T}) = (m+1)(n+1)f_2 - ((m+1)nf_1^{h,0} + (n+1)mf_1^{v,0}) + mnf_0^0$$

Theorem

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spline spaces $\mathcal{S}_{m,n}$ and higher dimensions

Corollary

For a given couple of integers $m, n \geqslant 1$, suppose that $\Omega \in \mathcal{A}^2_{m-1,n-1}$. Then, the basis of a space $\mathcal{S}_{m,n}(\mathcal{T})$ can be obtained as follows:
$\{b|_\Omega : b(x,y) \in \widehat{\mathcal{B}} \ \wedge \ \mathrm{supp}\, b(x,y) \cap \mathrm{int}\,\Omega \neq \emptyset\}$

Remark

The corollary above can be generalized for any dimension. Kraft's selection mechanism works fine for multivariate case generating a basis of a spline space over a hierarchical mesh.

some references

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Question

What is a structure?

candidates

fundamental groups, CMC (minimal) surfaces, metric,

conformal metric, symplectic structure, topology, smooth structures, complex structures, . . .

Would you accept new candidates?

THANK YOU FOR ◇ YOUR ATTENTION