Asymptotic Analysis of Partition Function $p(n)$

Thotsaporn "Aek" Thanatipanonda

Mahidol University, International College

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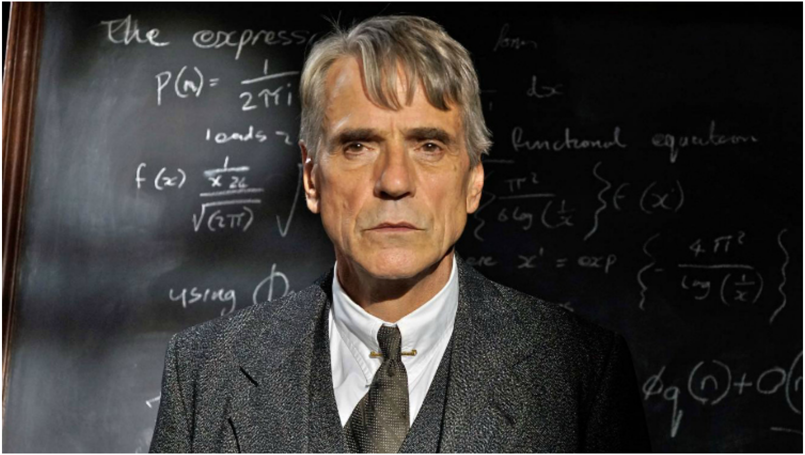
The man who knew infinity



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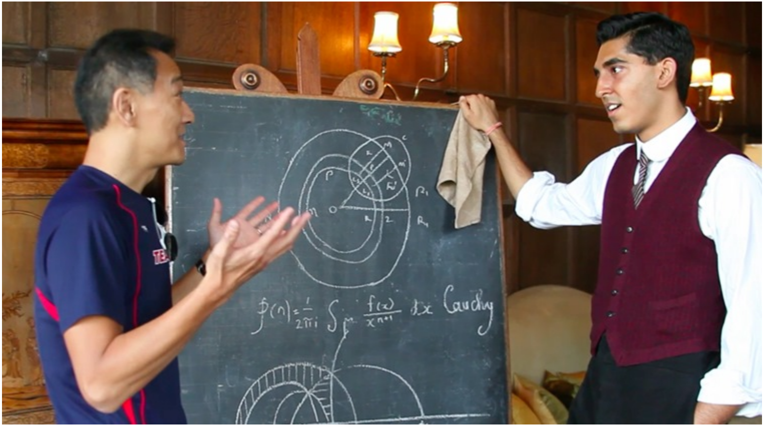

Ken Ono explaining Math to Dev Patel

Partition function $p(n)$

An integer partition is a way of writing n as a sum of positive integers.
Example: Let $n = 5$, n can also be written as $3 + 1 + 1$.

The number of integer partitions of n is given by the *partition function* $p(n)$.
Example: $p(5) = 7$ as we can write 5 in 7 different ways:

$$\begin{aligned} 5 &= 5 \\ &= 4 + 1 \\ &= 3 + 2 \\ &= 3 + 1 + 1 \\ &= 2 + 2 + 1 \\ &= 2 + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 1 + 1. \end{aligned}$$

Some Basics: Generating Function

Generating Function:

$$\begin{aligned}
P(z) &:= \sum_{n\geq 0} p(n)z^n \\
&= (1 + z + z^{1+1} + z^{1+1+1} + \dots)(1 + z^2 + z^{2+2} + \dots)\dots \\
&= \prod_{m=1}^{\infty} (1 + z^m + z^{2m} + z^{3m} + \dots) \\
&= \prod_{m=1}^{\infty} \left(\frac{1}{1 - z^m} \right).
\end{aligned}$$

Some Basics: Recurrence Relation of $p(n)$

Theorem (Euler's Theorem)

$$\begin{aligned} p(n) =& p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) \\ &+ p(n-15) - \dots \\ =& \sum_{j \geq 1} (-1)^{j+1} \left[p\left(n - \frac{3j^2 - j}{2}\right) + p\left(n - \frac{3j^2 + j}{2}\right) \right]. \end{aligned}$$

Hardy-Ramanujan Expansion of $p(n)$

An asymptotic expression for $p(n)$ is given by

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \qquad \text{as } n \to \infty.$$

This asymptotic formula was first obtained by G. H. Hardy and Ramanujan in 1918 and independently by J. V. Uspensky in 1920.

Hardy-Ramanujan Expansion of $p(n)$

Hardy and Ramanujan obtained an asymptotic expansion with the above approximation as the first term:

$$p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{v} A_k(n)\sqrt{k} \cdot \frac{d}{dx}\left(\frac{1}{\sqrt{x-\frac{1}{24}}} \exp\left[\frac{\pi}{k}\sqrt{\frac{2}{3}\left(x - \frac{1}{24}\right)} \right] \right)_{x=n},$$

where

$$A_k(n) = \sum_{0 \le m < k,\ (m,k)=1} e^{\pi i (s(m,k) - 2nm/k)}.$$

Rademacher's Better Approximation

In 1937, Hans Rademacher was able to improve on Hardy and Ramanujan's results by providing a convergent series expression for $p(n)$. It is

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \cdot \frac{d}{dx}\left(\frac{1}{\sqrt{x - \frac{1}{24}}} \sinh\left[\frac{\pi}{k}\sqrt{\frac{2}{3}\left(x - \frac{1}{24}\right)} \right] \right)_{x=n} .$$

Complex Analysis: Approximation at Pole

Given $f(z) = \sum_n a_n z^n$. How to find a good approximation of a_n?

Example 1:

$$f(z) = \frac{e^z}{1-z} \approx \frac{e}{1-z}.$$

Complex Analysis: Approximation at Pole

Example 2: Fibonacci sequence, F_n
Generating function:

$$\sum_{n\geq 0} F_n z^n = \frac{z}{1-z-z^2}.$$

The generating function gives an approximation of F_n

$$F_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

Complex Analysis: Approximation at Pole

Theorem

Let $f(z) = \sum a_n z^n$. Then

$$a_n \sim \left(\frac{1}{|z_0|}\right)^n$$

where z_0 is the closet singularity to the origin

Cauchy Theorem and The Saddle Point Bound

Theorem (Cauchy)

Let $f(z) = \sum a_n z^n$. *Then*

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}\, dz, \qquad n = 0, 1, 2, \ldots.$$

Theorem (Saddle-point Bounds)

$$a_n = [z^n] f(z) \leq \frac{\mathcal{M}(f;r)}{r^n},$$

where $\mathcal{M}(f;r) := \sup_{|z|=r} |f(z)|$.

Saddle Point Bounds: Example

Approximation of $n!$ via $f(z) = e^z$.

First Attempt: The Elementary Method

Theorem

$$p(n) \leq e^{\pi\sqrt{2n/3}(1+o(1))}.$$

Compare to Hardy-Ramanujan, this method is only off by the factor of n

Second Attempt: Mellin Transform

Theorem

$$p(n) \leq \frac{C}{n^{1/4}} e^{\pi\sqrt{2n/3}}.$$

Third Attempt: The Circle Method

Theorem (Hardy-Ramanujun (1918))

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \qquad \text{as } n \to \infty.$$