# Uniform Distribution Modulo One

## Theory and Application

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## Abstract

We discuss the distribution of fractional parts of real numbers in the unit interval. This concept was originated in 1916 by Hermann Weyl in his celebrated paper titled “Über die Gleichverteilung von Zahlen mod. Eins.” A surprising application of the theory is to show that the set of the Fibonacci numbers is extendable in any base.

# Uniform Distribution Modulo One

## Definition

The sequence $\omega = (x_n)$, $n = 1, 2, \ldots$, of real numbers is said to be **uniformly distributed modulo 1** (u.d. mod 1) if for every pair $a, b$ of real numbers with $0 \leq a < b \leq 1$ we have

$$\lim_{N \to \infty} \frac{A([a, b); N; \omega)}{N} = b - a,$$

where $A(E; N; \omega) = \mathrm{Card}\{x_n \,:\, 1 \leq n \leq N,\ \{x_n\} \in E\}$.

## Basic Properties

- The sequence $(x_n)$ is u.d. mod 1 if and only if for every complex-valued continuous function $f$ on $\mathbb{R}$ with period 1 we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\, dx.$$

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- If the sequence $(x_n)$ is u.d. mod 1, and if $(y_n)$ is a sequence with the property $\lim_{n\to\infty}(x_n - y_n) = \alpha$, a real constant, then $(y_n)$ is u.d. mod 1.

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- If $(x_n)$ is u.d. mod 1, then the sequence $(\{x_n\})$ of fractional parts is every dense in $\bar{I} = [0, 1]$. The converse is not true. For example, the sequence $(\{\sin n\})$ is dense in $\bar{I}$ but not u.d. mod 1. (For proof, see Appendix 1.)

# Uniform Distribution Modulo One

## Example

The sequence

$$\frac{0}{1}, \frac{0}{2}, \frac{1}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \ldots, \frac{0}{k}, \frac{1}{k}, \ldots, \frac{k-1}{k}, \ldots$$

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## Proof.

See Appendix 2. □

# Uniform Distribution Modulo One

Satz 1. *Gilt für jede ganze Zahl* $m \neq 0$ *die Limesgleichung*

$$\sum_{h=1}^{n} e(m\alpha_h) = o(n),$$

*so genügen die Zahlen* $\alpha_n$ mod. 1 *dem Gesetz der überall gleichmäßig dichten Verteilung.*

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**Theorem (Weyl Criterion)**

*The sequence $(x_n)$ is u.d. mod 1 if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \quad \text{for all integers } h \neq 0.$$

# Sequence of Multiples of an Irrational Number

Satz 2. *Ist* $\xi$ *eine irrationale Zahl, so liegen die ganzzahligen Vielfachen von* $\xi$:

$$1\xi,\ 2\xi,\ 3\xi,\ \cdots$$

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Ist $m$ eine ganze Zahl $\neq 0$ und setzen wir $m\xi = \eta$, so haben wir nur festzustellen, daß

$$\sum_{h=1}^{n} e(h\eta) = o(n)$$

wird. Die linke Seite läßt sich aber als geometrische Reihe summieren, ihr absoluter Betrag ist

$$= \left| \frac{e((n+1)\eta) - e(\eta)}{e(\eta) - 1} \right| \leq \frac{2}{|e(\eta) - 1|} = \frac{1}{|\sin \pi\eta|},$$

ist also, da $\eta$ keine ganze Zahl ist, nicht nur $= o(n)$, sondern bleibt sogar unterhalb einer endlichen Grenze.

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- The sequence $(p_n\theta)$ is u.d. mod 1, where $(p_n)$ is the sequence of primes arranged in increasing order. (Vinogradov, Hua)
- The sequence $(\omega(n)\theta)$ is u.d. mod 1, where $\omega(n)$ is the number of prime divisors of $n$. (Erdös, Delange)

# An Open Problem Concerning $(n\theta)$ by Erdös

For each positive integer $h$, let

$$A_h = \limsup_{N \to \infty} \left| \sum_{n=1}^{N} e^{2\pi i h n \theta} \right|.$$

Then $A_h$ is finite for every positive integer $h$. However, is it true that $\limsup_{h \to \infty} A_h$ is infinite?

# A Generalization

**Theorem**

*Let* $p(x) = \alpha_m x^m + \alpha_{m-1} x^{m-1} + \cdots + \alpha_0$, $m \geq 1$, *be a polynomial with real coefficients and let at least one of the coefficients* $\alpha_j$ *with* $j > 0$ *be irrational. Then the sequence* $(p(n))$, $n = 1, 2, \ldots,$ *is u.d. mod 1.*

# Application

## Definition

Suppose that $S$ is an infinite set of positive integers. We say that $S$ is **extendable in base** $b$ if for each positive integer $x$, there are positive integers $y$ and $n$, with $y < b^n$, such that $xb^n + y$ is in $S$.

# Application

**Theorem**

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## Proof.

1. The set $S = \{s_1, s_2, \ldots\}$ is extendable in base $b$ if and only if the sequence $(\{\log_b s_n\})$ of fractional parts is dense in $\bar{I}$. (See [1].)

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1. The set $S = \{s_1, s_2, \dots\}$ is extendable in base $b$ if and only if the sequence $(\{\log_b s_n\})$ of fractional parts is dense in $\bar{I}$. (See [1].)
2. If a sequence $(x_n)$ has the property that $\Delta x_n = x_{n+1} - x_n \to \theta$ (irrational) as $n \to \infty$, then the sequence $(x_n)$ is u.d. mod 1. (See Appendix 3.)

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2. If a sequence $(x_n)$ has the property that $\Delta x_n = x_{n+1} - x_n \to \theta$ (irrational) as $n \to \infty$, then the sequence $(x_n)$ is u.d. mod 1. (See Appendix 3.)
3. The sequence $(\log_b F_n)$ is u.d. mod 1 for every integer $b > 1$.

□

# References

[1] R. S. Bird, *Integers with given initial digits*, Amer. Math. Monthly, **79** (1972), 367–370.

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[4] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences* John Wiley & Sons, New York-London-Sydney, 1974.

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# Appendix 1

Let $x$ be a real number in $(0, 1)$. For each $\varepsilon > 0$, we want to show that there exists a $k$ such that $|x - \{x_k\}| < \varepsilon$. Without loss of generality, we may assume that $x - \varepsilon/2 \geq 0$ and $x + \varepsilon/2 \leq 1$. Since $(x_n)$ is u.d. mod 1, there exists an $N$ such that

$$\left| \frac{A([x - \varepsilon/2, x + \varepsilon/2); N)}{N} - \varepsilon \right| < \varepsilon.$$

This implies $A([x - \varepsilon/2, x + \varepsilon/2); N) > 0$. Hence there exists a $k$ $(1 \leq k \leq N)$ such that $x - \varepsilon/2 \leq \{x_k\} < x + \varepsilon/2$, so that

$$|x - \{x_k\}| < \varepsilon,$$

as desired. For the cases $x = 0$ and $x = 1$, we consider the intervals $[0, \varepsilon)$ and $[1 - \varepsilon, 1)$, respectively, instead.

# Appendix 2

Let $N$ be a positive integer and $0 \leq a < b \leq 1$. We try to compute $A([a,b);N)$. Let m be the largest positive integer such that $\frac{1}{2}m(m+1) \leq N.$ Then the first $N$ terms of the sequence can be listed as follows:

$$\frac{0}{1}, \frac{0}{2}, \frac{1}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \ldots, \frac{0}{m}, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, \frac{0}{m+1}, \frac{1}{m+1}, \ldots, \frac{\ell}{m+1},$$

where $0 \leq \ell < m$ if $\frac{1}{2}m(m+1) < N$. Now we partition this finite sequence into $m+1$ subpartitions by letting the $j$th subpartition ($1 \leq j \leq m$) consist of $\frac{0}{j}, \frac{1}{j}, \ldots, \frac{j-1}{j}$ and the last subpartition (could be empty if $\frac{1}{2}m(m+1) = N$) consist of $\frac{0}{m+1}, \frac{1}{m+1}, \ldots, \frac{\ell}{m+1}$.

# Appendix 2 (cont.)

Now we count the number of elements in the $j$th subpartition $(1 \leq j \leq m)$ that are in $[a, b)$, that is, we count the integers $k$ in $[0, j-1)$ such that $a \leq \frac{k}{j} < b$ or $ja \leq k < jb$. This number is $j(b-a) + \theta_j$ where $|\theta_j| < 1$. The number of elements in the last subpartition is at most $m$. Thus,

$$\sum_{j=1}^{m} j(b-a) + \theta_j \leq A([a, b); N) \leq \sum_{j=1}^{m} j(b-a) + \theta_j + m.$$

Since $-1 < \theta_j < 1$ and $\sum_{j=1}^{m} j = \frac{1}{2}m(m+1)$, we have further that

$$\frac{1}{2}(b-a)m(m+1)-m < A([a, b); N) < \frac{1}{2}(b-a)m(m+1)+m+m.$$

# Appendix 2 (cont.)

From the way the integer $m$ is defined, we have $\frac{1}{2}m(m+1) \leq N < \frac{1}{2}(m+1)(m+2)$, so that,

$$(b-a)(N-(m+1)) - m < A([a,b);N) < (b-a)m + 2m,$$

or, equivalently,

$$(b-a) - \frac{b-a}{N} - \frac{m}{N}(1+b-a) < \frac{A([a,b);N)}{N} < (b-a) + \frac{2m}{N}.$$

The integer $m$ can be approximated by $\frac{1}{2}m^2 < N < \frac{1}{2}(m+2)^2$ or $\sqrt{2N} - 2 < m < \sqrt{2N}$, so that $\frac{\sqrt{2N}-2}{N} < \frac{m}{N} < \frac{\sqrt{2N}}{N}$. Consequently, $\frac{m}{N} \to 0$ as $N \to \infty$.

# Appendix 2 (cont.)

Combining with the last inequality that approximates $\frac{A([a,b);N)}{N}$, we have

$$\lim_{N\to\infty} \frac{A([a,b);N)}{N} = b - a,$$

as desired.

# Appendix 3

Let $q$ be a positive integer, then since $\Delta x_n \to \theta$ as $n \to \infty$, there exists an integer $g_0 = g_0(q)$ such that for any integers $n > g \geq g_0$, $|\Delta x_j - \theta| \leq \frac{1}{q^2}$ for $j = g, g+1, \ldots, n-1$. Since $\sum_{j=a}^{b-1} \Delta x_j = x_b - x_a$ for $1 \leq a < b$, we have

$$|x_n - x_g - (n-g)\theta| = \left| \sum_{j=g}^{n-1} (\Delta x_j - \theta) \right| \leq \sum_{j=g}^{n-1} |\Delta x_j - \theta| \leq \sum_{j=g}^{n-1} \frac{1}{q^2} = \frac{n-g}{q^2}.$$

For arbitrary real numbers $u$ and $v$, we have

$$\begin{aligned} |e^{2\pi iu} - e^{2\pi iv}| &= \left| e^{\pi i(u+v)} (e^{\pi i(u-v)} - e^{-\pi i(u-v)}) \right| \\ &= \left| 2i \sin \pi(u-v) \right| \leq 2\pi |u-v|. \end{aligned}$$

# Appendix 3 (cont.)

Hence, if $h \neq 0$ is an integer, then

$$e^{2\pi i h x_n} - e^{2\pi i h (x_g + (n-g)\theta)} \leq \frac{2\pi |h| (n-g)}{q^2}.$$

# Appendix 3 (cont.)

By the triangle inequality, it follows that

$$\left|\sum_{n=g}^{g+q-1} e^{2\pi i h x_n}\right| \le \left|\sum_{n=g}^{g+q-1} e^{2\pi i h (x_g+(n-g)\theta)}\right| + \frac{2\pi|h|}{q^2}\sum_{n=g}^{g+q-1}(n-g) \le K,$$

where $K = |\sin \pi h\theta|^{-1} + \pi|h|$. Thus, by induction, for every positive integer $H$,

$$\left|\sum_{n=g}^{g-1+Hq} e^{2\pi i h x_n}\right| \le HK.$$

# Appendix 3 (cont.)

Let $N \geq g$ be an integer. Choose the largest integer $H$ such that $g - 1 + Hq < N$. It follows that

$$g - 1 + Hq \leq N - 1 \implies H \leq \frac{N - g}{q}$$

and

$$g - 1 + (H + 1)q \geq N \implies N - g - Hq + 1 \leq q.$$

# Appendix 3 (cont.)

We have

$$\left|\sum_{n=1}^{N} e^{2\pi i h x_n}\right| = \left|\sum_{n=1}^{g-1} e^{2\pi i h x_n} + \sum_{n=g}^{g-1+Hq} e^{2\pi i h x_n} + \sum_{n=g+Hq}^{N} e^{2\pi i h x_n}\right|$$
$$\leq (g-1) + HK + (N-g-Hq+1)$$
$$\leq g-1+\frac{N-g}{q}K+q.$$

Keeping $q$ fixed, we obtain

$$\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n}\right| \leq \frac{K}{q}.$$

Since $q$ can be as large as we please, the theorem follows.