Weierstrass $\wp$-Function

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Outline

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Abstract

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. The Weierstrass $\wp$-function is defined by the series

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

We consider it to be the quintessential elliptic function from which all others can easily be obtained. In particular, it can be shown that any elliptic function is a rational function of $\wp$ and $\wp'$. In this talk we give an overview of the Weierstrass $\wp$-function including its algebraic properties and connections with elliptic curves.

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  2. f is meromorphic (its only singularities in the finite plane are poles).
- **Theorem 1** A nonconstant elliptic function has a fundamental pair of periods.
- **Theorem 2** If an elliptic function f has no poles in some period parallelogram, then f is constant.

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- **Theorem 4** The contour integral of an elliptic function taken along the boundary of any cell is zero.

- **Theorem 5** The sum of the residues of an elliptic function at its poles in any period parallelogram is zero.
- **Theorem 6** The number of zeros of an elliptic function in any period parallelogram is equal to the number of poles, each counted with multiplicity.

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- **Theorem 7** The function $\wp$ so defined has periods ω_1 and ω_2. It is analytic except for a double pole at each period $\omega \in \Lambda$. Moreover $\wp(z)$ is an even function of z.

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Differential Equation Satisfied by $\wp$

- **Theorem 8** The function $\wp$ satisfies the nonlinear differential equation
$$\big(\wp'(z)\big)^2 = 4\wp^3(z) - 60G_4\wp(z) - 140G_6.$$

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The Numbers e_1, e_2, e_3

- We denote by e_1, e_2, e_3 the values of $\wp$ at the half-periods:

$$e_1 = \wp\left(\frac{\omega_1}{2}\right), \quad e_2 = \wp\left(\frac{\omega_2}{2}\right), \quad e_3 = \wp\left(\frac{\omega_1 + \omega_2}{2}\right).$$

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- **Theorem 9** We have

$$4\wp^3(z) - g_2\wp(z) - g_3 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

Moreover, the roots e_1, e_2, e_3 are distinct, hence $g_2^3 - 27g_3^2 \neq 0$.

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Weierstrass $\wp$-Function vs. Elliptic Functions

- **Theorem 10** Any even elliptic function (relative to periods ω_1 and ω_2) is a rational function of $\wp(z)$. Any elliptic function f is of the form

$$f(z) = g\big(\wp(z)\big) + \wp'(z)h\big(\wp(z)\big)$$

with g and h rational.

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- The group law is constructed geometrically.
- Elliptic curves can have points with coordinates in any field, such as $\mathbb{F}_p, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

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- We can use geometry to make the points of an elliptic curve into a group.

- **Theorem 11** (Poincaré, around 1900) Let K be a field and suppose that an elliptic curve E is given by an equation of the form

$$E:\, y^2 = x^3 + Ax + B \quad \text{with } A, B \in K.$$

Let $E(K)$ denote the set of points of E with coordinates in K,

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- What does $E(K)$ look like?

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- The quotient $\mathbb{C}/\Lambda$ is both a group and a complex manifold.
- There is a complex analytic isomorphism from $\mathbb{C}/\Lambda$ to $E(\mathbb{C})$ given by

$$z \mapsto \begin{cases} \left(\wp(z), \frac{1}{2}\wp'(z)\right) & \text{if } z \notin \Lambda, \\ \mathcal{O} & \text{if } z \in \Lambda. \end{cases}$$

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- Hence

$$E(\mathbb{C}) \cong \mathbb{C}/\Lambda \cong S^1 \times S^1,$$

i.e., $E(\mathbb{C})$ can be visualized as a torus.

Thank you!!