Well-Quasi-Ordering by Graph Relations

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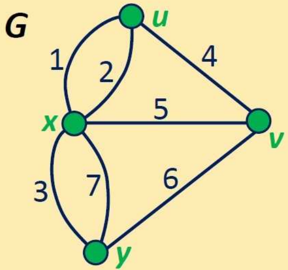
June 14, 2017

Well-Quasi-Ordering (WQO)

- $(X, \leqslant)$: A binary relation $\leqslant$ defined on a set X.
- $(X, \leqslant)$ is a **quasi-ordering** if $\leqslant$ is reflexive and transitive.
- $(X, \leqslant)$ is **well-quasi-ordering (WQO)** if every infinite sequence $x_0, x_1, \ldots$ in X there exist $i < j$ such that $x_i \leqslant x_j$. **"GOOD PAIR"**
- $(X, \leqslant)$ is WQO iff it contains neither **an infinite antichain (no two elements comparable)** nor an infinite strictly decreasing sequence.

Graphs

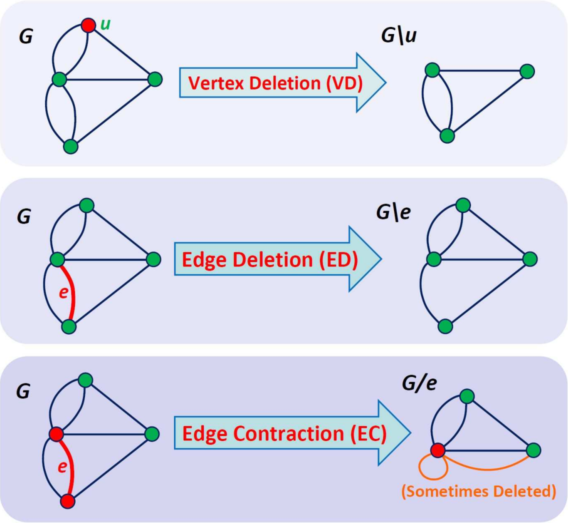


A graph $G = (V, E)$

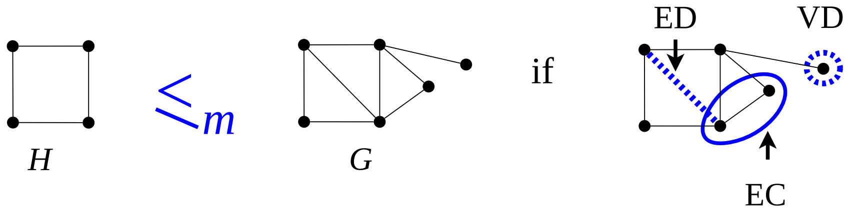
$V = \{u, v, x, y\}$

$E = \{1, 2, 3, 4, 5, 6, 7\}$

Graph Operations



Minor Relation ($\leqslant_m$)



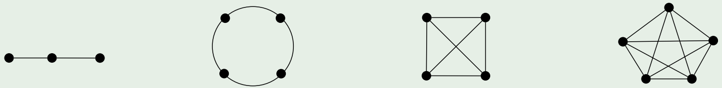
a (possibly empty) sequence of

- vertex deletion (VD),
- edge deletion (ED), and
- edge contraction (EC).

Containment Relation	VD	ED	EC
Minor	Yes	Yes	Yes
Induced Minor	Yes	No	Yes
Subgraph	Yes	Yes	No
Induced Subgraph	Yes	No	No

Vertex Deletion (VD), Edge Deletion (ED), Edge Contraction (EC)

Example



P_3 C_4 K_4 K_5

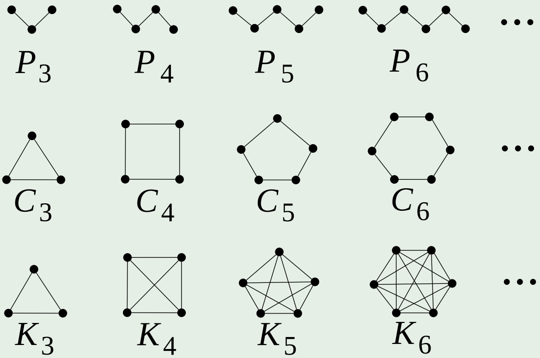
Graph Minor Theorem

Robertson and Seymour (2004)

The Graph Minor Theorem (Wagner's Conjecture)

(finite graphs, $\leqslant_m$) is WQO.

Example



Other Graph Relations

Other graph-containment relations?

Containment Relation	**VD**	**ED**	**EC**
Minor	Yes	Yes	Yes
Induced Minor	Yes	No	Yes
Subgraph	Yes	Yes	No
Induced Subgraph	Yes	No	No

Vertex Deletion (VD), Edge Deletion (ED), Edge Contraction (EC)

Subgraph and Induced Subgraph

Subgraph (VD + ED) and **Induced Subgraph** (VD)

- Is (finite graphs, $\leqslant_s$) WQO?
- Infinite antichain?

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Subgraph ($\leqslant_s$, VD + ED) and **Induced Subgraph** ($\leqslant_{is}$,VD)

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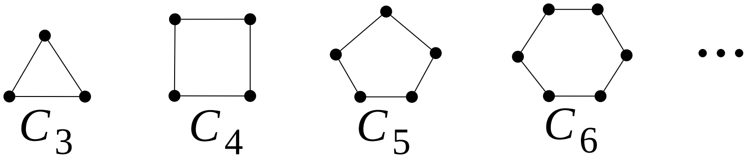

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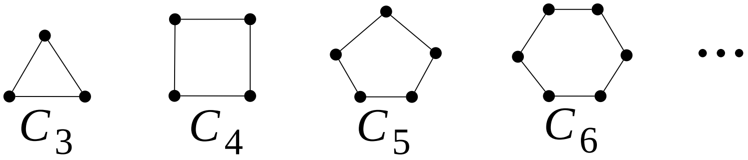


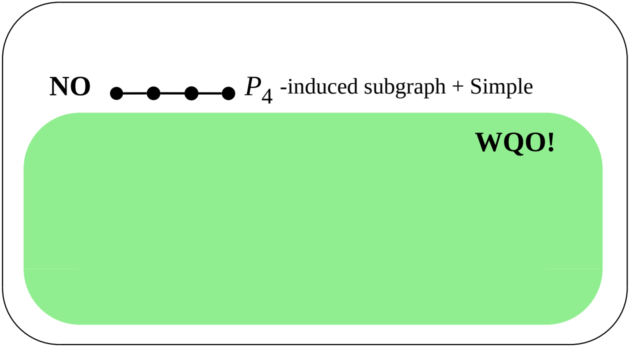
- Is (finite graphs, $\leqslant_{is}$) WQO?

Subgraph and Induced Subgraph

Theorem (Damaschke '90)

(no P_4-induced subgraph + Simple, $\leqslant_{is}$) is WQO.

The class of all finite graphs



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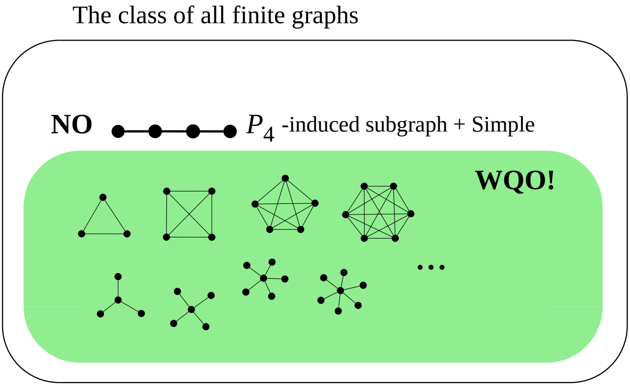


Subgraph and Induced Subgraph

Theorem (Ding '92)

Let n be a positive integer. (no P_n-subgraph + Simple, $\leqslant_{is}$) is WQO.

$\mathcal{I}$: an ideal with respect to $\leqslant_s$ if $H \leqslant_s G \in \mathcal{I} \Rightarrow H \in \mathcal{I}$.

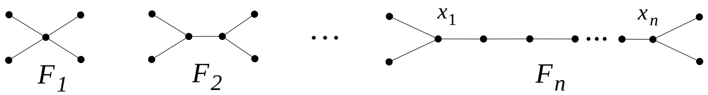
Theorem (Ding '92)

The following are equivalent:

(i) $\mathcal{I}$ is wqo by subgraph;

(ii) $\mathcal{I}$ is wqo by induced subgraph;

(iii) $\mathcal{I}$ contains only finitely many graphs C_n and F_n.



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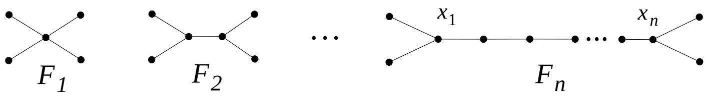


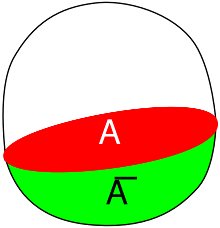
Well-Quasi-Ordering (WQO)

A: An antichain of $(X, \leqslant)$

- $\bar{A} = \{g \in X : g < h \text{ for some } h \in A\}$
- $\bar{A}$ is **fundamental** if $\bar{A}$ has no infinite antichains.

Lemma (Ding '09)

$(X, \leqslant)$ has an infinite antichain $\Rightarrow$ there is a fundamental infinite antichain A.



Induced Minor ($\leqslant_{im}$, VD + EC)

- Is (finite graphs, $\leqslant_{im}$) WQO?
- Infinite antichain?

Induced Minor

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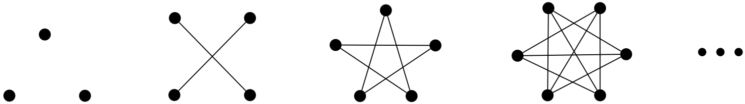
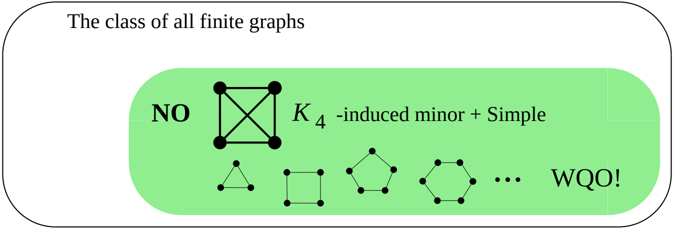


(The Complement of C_n)

Induced Minor

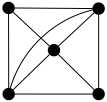
Theorem (Thomas '85)

(no K_4-induced minor + Simple, $\leqslant_{im}$) is WQO.



Problem: Is (no $K_5 \backslash e$-induced minor + Simple, $\leqslant_{im}$) is WQO?

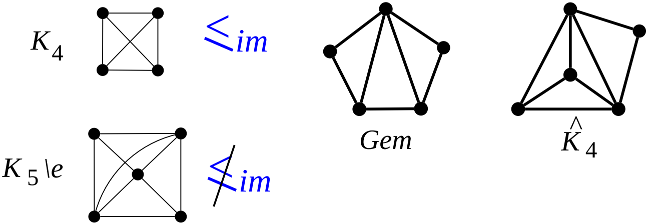
$K_5 \backslash e$



Induced Minor

Theorem (Blasiok, Kamiński, Raymond, Trunk '15)

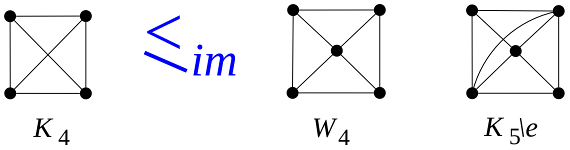
(no H-induced minor, $\leqslant_{im}$) is WQO iff H $\leqslant_{im}$ Gem or $\hat{K}_4$.

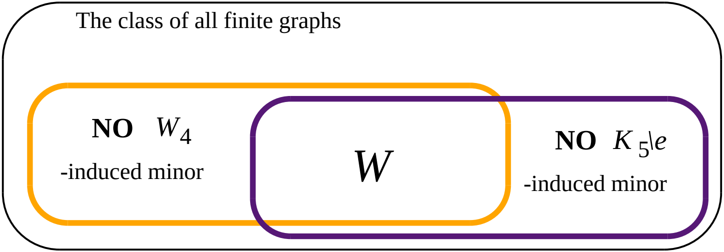


Remark: Answer Thomas' problem.

Induced Minor

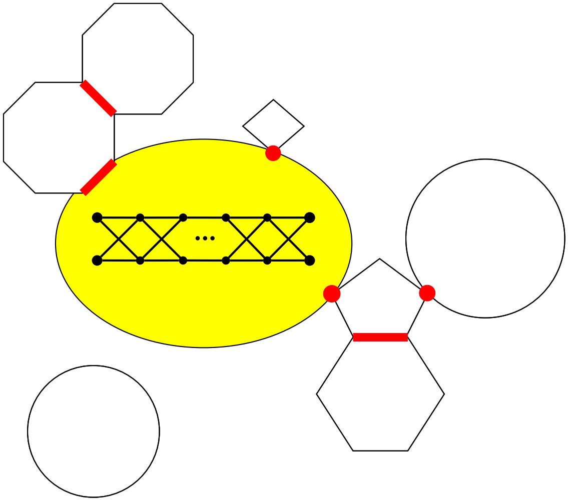
$\mathcal{W}$: The class of graphs without W_4 and $K_5\setminus e$ as an induced minors.



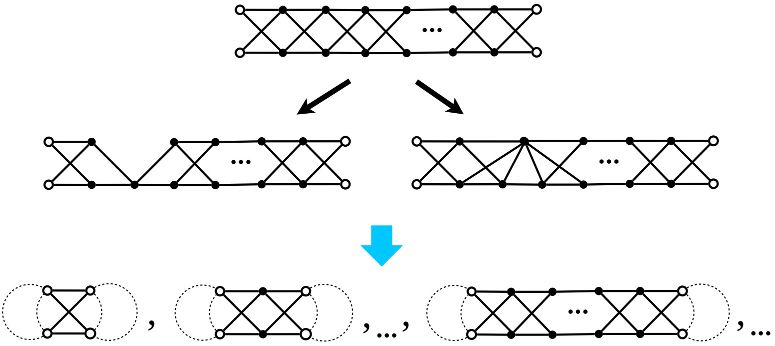


Induced Minor

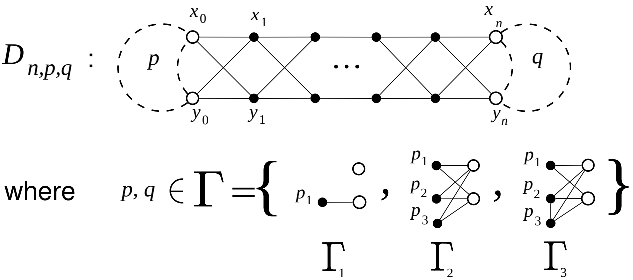
$(\mathcal{W}, \leqslant_{im})$ is not WQO (there is an infinite antichain).



Induced Minor

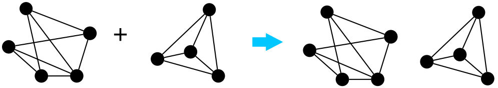


- $(\mathcal{W}, \leqslant_{im})$ is not wqo (there is an infinite antichain).
- Infinite Antichain $\mathcal{D}^{\Gamma}$.

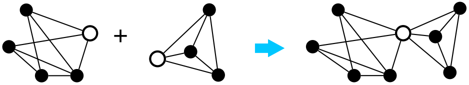


Induced Minor

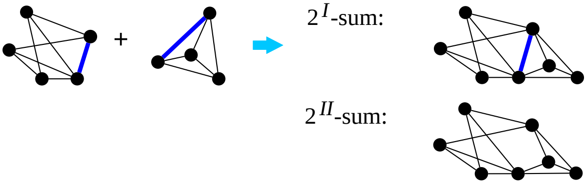
- 0-sum:



- 1-sum:

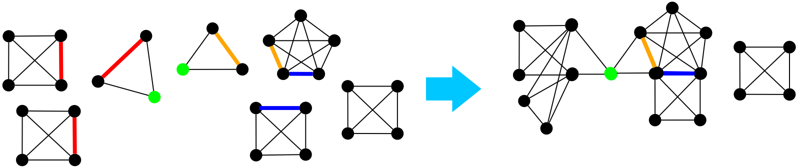


- 2-sum:



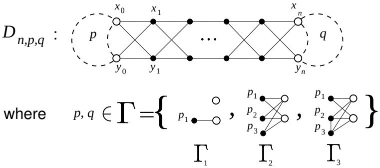
Induced Minor

$\mathcal{S}$ = { 0-, 1-, 2-sums of K_n : $2^{//}$-sum is performed over edge e only when every K_n containing e is K_3 or K_4 }



Induced Minor

- $\mathcal{W} = \{W_4, K_5 \backslash e\}_{im}$−free $=$ { 0-, 1-, 2-sums of cliques }.
- Infinite Antichain, $\mathcal{D}^{\Gamma}$.



$\Rightarrow$ $\{W_4, K_5 \backslash e\}_{im}$−free is not wqo by $\leqslant_{im}$.
$\Rightarrow$ $\{K_5 \backslash e\}_{im}$−free is not wqo by $\leqslant_{im}$. (Thomas' proposed problem)

Theorem

Let $\mathcal{Z} \subseteq \mathcal{W}$. $(\mathcal{Z}, \leqslant_{im})$ is wqo iff $\mathcal{Z} \cap \mathcal{D}^{\Gamma}$ is finite.

$\Rightarrow$ $\{K_4\}_{im}$−free is wqo by $\leqslant_{im}$. (Thomas' result)

Thank You!