Arithmetic dynamics with good reduction

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Dynamics on $\mathbb{P}^1$

Let $\phi \in \mathbb{Q}(z)$ be a rational function of degree $d \geq 2$.

$$\deg \phi := \max\{\deg f_1, \deg f_2\}, \text{ where } \phi = f_1/f_2 \text{ in lowest terms.}$$

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Definition

A point $z \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \bigcup \{\infty\}$ is called **preperiodic** if

$$\phi^n(z) = \phi^m(z) \text{ for some } n > m \geq 0$$

Write $\mathrm{PrePer}(\phi, \mathbb{Q}) := \{z \in \mathbb{P}^1(\mathbb{Q}) : z \text{ is preperiodic under } \phi\}$

Parametrization of period 4

Theorem (Netto, Morton, Erkama)

Let $\phi(z) = z^2 + c$. Then c and the points of period 4 can be parametrized over $\mathbb{C}$ by the following:

$$c = \frac{1 - 4t^3 - t^6}{4t^2(t^2 - 1)},$$

$$x_1, x_3 = \frac{t^4 - t^2 \pm \sqrt{(t^4 - 1)(t^2 + 2t - 1)}}{2t(t^2 - 1)},$$

$$x_2, x_4 = \frac{1 - t^2 \pm t\sqrt{(t^4 - 1)(t^2 + 2t - 1)}}{2t(t^2 - 1)}, t = x_1 + x_3$$

Points of period 4 of $\phi(z) = z^2 + c$

Example $\phi(x) = x^2 - \frac{31}{48}$

$$
\begin{array}{ccc}
\frac{1}{4} + \frac{\sqrt{-15}}{6} & \longrightarrow & -1 + \frac{\sqrt{-15}}{12} \\
\uparrow & \phi & \downarrow \\
-1 - \frac{\sqrt{-15}}{12} & \longleftarrow & \frac{1}{4} - \frac{\sqrt{-15}}{6}
\end{array}
$$

Points of period 6 over quadratic field

Is it true for period 6?

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Example(Flynn-Poonen-Schaefer, 1997) $\phi(x) = x^2 - \frac{71}{48}$

$$-1+\frac{\sqrt{33}}{12} \xrightarrow{\phi} -\frac{1}{4}-\frac{\sqrt{33}}{6} \xrightarrow{\phi} -\frac{1}{2}+\frac{\sqrt{33}}{12} \xrightarrow{\phi} -1-\frac{\sqrt{33}}{12} \xrightarrow{\phi} -\frac{1}{4}+\frac{\sqrt{33}}{6} \xrightarrow{\phi} -\frac{1}{2}-\frac{\sqrt{33}}{12} \xrightarrow{\phi} -1+\frac{\sqrt{33}}{12}$$

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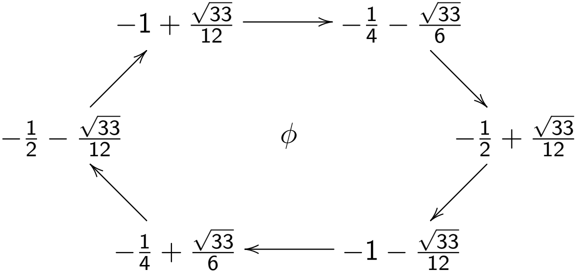

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- the Birch and Swinnerton-Dyer conjecture holds (used by Stoll, 2008).

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 & -1 + \frac{\sqrt{33}}{12} & \longrightarrow & -\frac{1}{4} - \frac{\sqrt{33}}{6} & \\
\nearrow & & & & \searrow \\
-\frac{1}{2} - \frac{\sqrt{33}}{12} & & \phi & & -\frac{1}{2} + \frac{\sqrt{33}}{12} \\
\nwarrow & & & & \swarrow \\
 & -\frac{1}{4} + \frac{\sqrt{33}}{6} & \longleftarrow & -1 - \frac{\sqrt{33}}{12} &
\end{array}$$

The above is the **only cycle** in all quadratic fields if

- the Birch and Swinnerton-Dyer conjecture holds (used by Stoll, 2008).
- $\overline{x}_i = x_{i+3}$, for all $\phi(x) = x^2 + c$ with $c \in \mathbb{Q}$.

Let p be a prime number and $x = \frac{a}{b}p^n \in \mathbb{Q}$ where a, b, p are relatively prime.
The $p-$adic norm of Q is defined as $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$
$|x|_p = p^{-n}$ $(|0|_p = 0)$.

Units of p-adic Norm (Place)

p-adic norm

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Note that $|\cdot|_p$ is a norm.
Moreover $|x+y|_p \leq \max(|x|_p, |y|_p) \leq |x|_p + |y|_p$ (it is non-archimedean).

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$R_p^* = \{\alpha \in \mathbb{Q} : |\alpha|_p = 1\}$, the group of units.

Definition

Let $\phi \in \mathbb{Q}(z)$ a rational function.
We say ϕ has **good reduction** at p if ϕ makes sense modulo v, i.e., if ϕ may be written as

$$\phi(\frac{x}{y}) = \frac{F_1(x,y)}{F_2(x,y)}$$

with $F_1, F_2 \in \mathbb{Z}[x,y]$ homogeneous of the same degree such that the reductions $\overline{F}_1$ and $\overline{F}_2$ modulo v have no common zeros besides $(0,0)$.

Example

$\phi(z) = z^2 - \frac{133}{144} \in \mathbb{Q}[z]$. We can write

$$\phi(\frac{x}{y}) = \frac{144x^2 - 133y^2}{144y^2}$$

For $p \geq 5$, ϕ has good reduction at p.
But reducing mod 2 and 3 gives:

$$\phi = \frac{0x^2 + y^2}{0y^2},$$

which is $0/0$ at $[x, y] = [1, 0]$. So ϕ has bad reduction at $2, 3$.

Dynatomic units

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Theorem (Morton and Silverman, 1995)

Let $\phi \in K(z)$ be a rational map of degree $d \geq 2$ with good reduction. Let $n_1, n_2 \in \mathbb{Z}$ be integers with $n_1 \nmid n_2$ and $n_2 \nmid n_1$, let $P_1, P_2 \in \mathbb{P}^1(K)$ be periodic points of exact periods n_1 and n_2, respectively, and write $P_i = [x_i, y_i]$ in normalized form. Then $x_1 y_2 - x_2 y_1 \in R^*$.

Cross ratios

Definition

Let $P_1, P_2, P_3, P_4 \in \mathbb{P}^1(K)$, and choose homogeneous coordinates $P_i = [x_i, y_i]$ for each point. The cross-ratio of P_1, P_2, P_3, P_4 is the quantity

$$\kappa(P_1, P_2, P_3, P_4) = \frac{(x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2)}{(x_1y_2 - x_2y_1)(x_3y_4 - x_4y_3)}.$$

Theorem (Morton and Silverman, 1995)

Let $\phi \in K(z)$ be a rational map of degree $d \geq 2$ with good reduction. Let $P \in \mathbb{P}^1(K)$ be a periodic point for ϕ of exact period n, and let i and j be integers satisfying

$$\gcd(i, n) = \gcd(j - 1, n) = \gcd(i - j, n) = 1.$$

Then

$$\kappa(P, \phi(P), \phi^i(P), \phi^j(P)) \in R^*.$$

Theorem (Canci-Paladino, 2016)

Let $\phi : \mathbb{P}^1(\mathbb{Q}) \longrightarrow \mathbb{P}^1(\mathbb{Q})$ *be an endomorphism defined over* $\mathbb{Q}$, *with good reduction at every non-archimedean place.*

- *If* $P \in \mathbb{P}^1(\mathbb{Q})$ *is a periodic point for* ϕ *with minimal period* n, *then* $n \leq 3$.
- *If* $P \in \mathbb{P}^1(\mathbb{Q})$ *is a preperiodic point for* ϕ, *then* $|\mathcal{O}_\phi(P)| \leq 12$.

Thank you.