# Unit 1
# Limits

1. The concept of **limits** involves the notion of getting closer and closer to something, but yet not touching it.

2. We will let a variable "inch up" to a particular value and examine the effect it has on the values of a function.

**Definition 1** **The limit of $f(x)$ as $x$ approaches $a$** is the number $L$, written

$$\lim_{x\to a} f(x) = L$$

provided that $f(x)$ is arbitrarily close to $L$ for all $x$ sufficiently close to, but not equal to, $a$. If there is no such number, we say that the limit **does not exist**.

3. **Properties of Limits**

   (a) If $f(x) = c$ is a constant function, then
   $$\lim_{x\to a} f(x) = \lim_{x\to a} c = c.$$

   (b) $\lim_{x\to a} x^n = a^n$, for any positive integer $n$

   If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

   (c) $\lim_{x\to a} \big(f(x) \pm g(x)\big) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$.
   *That is, the limit of a sum or difference is the sum or difference, respectively, of the limits.*

   (d) $\lim_{x\to a} \big(f(x) \cdot g(x)\big) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.
   *That is, the limit of a product is the product of the limits.*

   (e) $\lim_{x\to a} \big(cf(x)\big) = c \cdot \lim_{x\to a} f(x)$, where $c$ is a constant.
   *That is, the limit of a constant times a function is the constant times the limit of the function.*

   R If $f$ is a polynomial function, then
   $$\lim_{x\to a} f(x) = f(a).$$

   (f) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$.
   *That is, the limit of a quotient is the quotient of limits, provided that the denominator does not have a limit of* 0.

   (g) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$.
   *For this property, if $n$ is even, we require that $\lim_{x\to a} f(x)$ be positive.*

   (h) if $f$ and $g$ are two functions for which $f(x) = g(x)$, for all $x \neq a$, then
   $$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

**Problem 1** Find the limits.

(a) $\lim_{x\to -2} \frac{x^2+2x}{x+2}$ (b) $\lim_{x\to 2} \frac{x^3-8}{x-2}$ (c) $\lim_{x\to 1} \frac{x-\frac{1}{x}}{1-\frac{1}{x}}$

(d) $\lim_{x\to -1} \frac{x+1}{x+1}$ (e) $\lim_{x\to -3} \frac{x^4-81}{x^2+8x+15}$ (f) $\lim_{x\to 2} \frac{3x^2-x-10}{x^2+5x-14}$

**Problem 2** Find

$$\lim_{x\to 6} \frac{\sqrt{x-2}-2}{x-6}.$$

**Problem 3** Find the constant $c$ so that

$$\lim_{x\to 3} \frac{x^2+x+c}{x^2-5x+6}$$

exists. For this value of $c$, determine the limit.

5. A special limit:

$$\lim_{x\to 0}(1+x)^{1/x} = e \approx 2.718.$$

**Exercise 1** Problems 10.1: 2, 4, 16, 24, 32, 34, 42, 45 ■

# Unit 2

# Limits (continued)

1. Terms to consider: one-sided limits, infinite limits, limits at infinity.

2. The limit exists if and only if both one-sided limits exist and are equal.

**Problem 4 — One-Sided Limits and Infinite Limits.** Find the limit (if it exists).

(a) $\displaystyle\lim_{x\to -1^+} \frac{2}{x+1}$

(b) $\displaystyle\lim_{x\to 2^-} \frac{2-x-x^2}{x^2-4}$

**Problem 5 — Limits at Infinity.** Find the limit (if it exists).

(a) $\displaystyle\lim_{x\to \infty} \frac{4}{(x-5)^3}$

(b) $\displaystyle\lim_{x\to -\infty} \sqrt{4-x}$

3. We note the following:

$$\lim_{x\to\pm\infty} \frac{1}{x^p} = 0 \qquad \text{where } p > 0.$$

**Theorem 1 — Limits at Infinity for Rational Functions.** If $f(x)$ is a rational function and $a_n x^n$ and $b_m x^m$ are the terms in the numerator and denominator, respectively, with the greatest powers of $x$, then

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}.$$

**Problem 6** Find the limit (if it exists).

(a) $\displaystyle\lim_{x\to \infty} \frac{x^2-1}{7-2x+8x^2}$

(b) $\displaystyle\lim_{x\to -\infty} \frac{x}{(3x-1)^2}$

(c) $\displaystyle\lim_{x\to \infty} \frac{x^5-x^4}{x^4-x^3+2}$

5. As $x \to \infty$ (or $x \to -\infty$), the limit of a polynomial function is the same as the limit of its term that involves the greatest power of $x$. For example, $\lim_{x\to-\infty}(x^3-x^2+x-2) = \lim_{x\to-\infty} x^3 = -\infty$.

6. **Limits for a Case-Defined Function**

**Problem 7** If $f(x) = \begin{cases} x^2+1 & \text{if } x \geq 1 \\ 3 & \text{if } x < 1 \end{cases}$, find the limit (if it exists):

$$\lim_{x\to 1^+} f(x), \qquad \lim_{x\to 1^-} f(x), \qquad \lim_{x\to 1} f(x), \qquad \lim_{x\to\infty} f(x), \qquad \lim_{x\to -\infty} f(x).$$

**Exercise 2** Problems 10.2: 2, 4, 8, 16, 18, 44, 52, 58, 64 ■

# Unit 3

## Continuity

**Definition 2** A function $f$ is **continuous** at $a$ if and only if the following three conditions are met:

(a) $f(a)$ exists.

(b) $\lim_{x\to a} f(x)$ exists.

(c) $\lim_{x\to a} f(x) = f(a)$.

2. A polynomial function is continuous at every point.

3. **Discontinuities of a Rational Function**

**Theorem 2** A rational function is discontinuous at points where the denominator is 0 and is continuous otherwise. Thus, a rational function is continuous on its domain.

**Problem 8** Determine whether the function $f(x) = \dfrac{x-4}{x^2-16}$ is continuous at 4 and at $-4$.

**Problem 9** Determine whether the function $f(x) = \begin{cases} x+2 & \text{if } x \geq 2 \\ x^2 & \text{if } x < 2 \end{cases}$ is continuous at 2 and at 0.

**Problem 10** Find all points of discontinuity of each function:

(1) $f(x) = \dfrac{x^2+3x-4}{x^2-4}$

(2) $f(x) = \dfrac{x^4}{x^4-1}$

(3) $f(x) = \begin{cases} \frac{16}{x^2} & \text{if } x \geq 2 \\ 3x-2 & \text{if } x < 2 \end{cases}$

**Exercise 3** Problems 10.3: 2, 9, 12, 20, 30 ■

# Unit 4

# The Derivative

1. A **secant line** is a line that intersects a curve at two or more points.
2. The **tangent line** to the curve at $P$ is defined to be the common limiting position of the secant lines joining the point $P$ with any other points of the curve.
3. The **slope of a curve** at a point $P$ is the slope, if it exists, of the tangent line at $P$.

**Definition 3** The slope of the tangent line at $\big(a, f(a)\big)$ is given by

$$m_{\tan} = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

**Problem 11** Find the slope of the tangent line to the curve $y = f(x) = x^2$ at the point $(1, 1)$.

**Definition 4** The **derivative** of a function $f$ is the function denoted $f'$ and defined by

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that this limit exists. If $f'(a)$ can be found, then $f$ is said to be **differentiable** at $a$, and $f'(a)$ is called the derivative of $f$ at $a$ or the derivative of $f$ with respect to $x$ at $a$. The process of finding the derivative is called **differentiation**.

6. Because the derivative gives the slope of the tangent line, $f'(a)$ is the slope of the line tangent to the graph of $y = f(x)$ at $\big(a, f(a)\big)$.
7. If $f$ is differentiable at $a$, then $f$ is continuous at $a$. That is, differentiability at a point implies continuity at that point.
8. It is false that continuity implies differentiability. For example, consider the function $f(x) = |x|$. This function is continuous at 0 but not differentiable there.

**Problem 12** Use the definition of the derivative to find each of the following: (1) $f'(x)$ if $f(x) = 4x - 1$ (2) $\dfrac{dp}{dq}$ if $p = 3q^2 + 2q + 1$ (3) $\dfrac{d}{dx}\sqrt{x+2}$

**Problem 13** Find an equation of the tangent line to the curve $y = \dfrac{3}{x-1}$ at the point $(2, 3)$.

**Problem 14** Find an equation of the tangent line to the curve $y = (x-7)^2$ at the point $(6, 1)$.

**Exercise 4** Problems 11.1: 4, 8, 10, 12, 16, 18, 22, 28 ∎

# Unit 5

## Rules for Differentiation

**Rule 1** Derivative of a Constant: If $c$ is a constant, then

$$\frac{d}{dx}(c) = 0.$$

That is, the derivative of a constant function is zero.

**Rule 2** Derivative of $x^n$: If $n$ is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

That is, the derivative of a constant power of $x$ is the exponent times $x$ raised to a power one less than the given power.

**Rule 3** Constant Factor Rule: If $f$ is a differentiable function and $c$ is a constant, then $cf(x)$ is differentiable, and

$$\frac{d}{dx}\big(cf(x)\big) = cf'(x).$$

That is, the derivative of a constant times a function is the constant times the derivative of the function.

**Rule 4** Sum or Difference Rule: If $f$ and $g$ are differentiable functions, then $f + g$ and $f - g$ are differentiable, and

$$\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x).$$

That is, the derivative of the sum (difference) of two functions is the sum (difference) of their derivatives.

**Problem 15** Differentiate the functions.

(a) $f(x) = \sqrt[3]{x}(\sqrt[4]{x} - 6x + 3)$ (b) $f(x) = \dfrac{x}{7} + \dfrac{7}{x}$

(c) $f(x) = x^2(x-2)(x+4)$ (d) $f(x) = \dfrac{\sqrt[3]{8x^2} + x}{6\sqrt{x}}$

**Problem 16** Find an equation of the tangent line to the curve.

(a) $y = 3 + x - 5x^2 + x^4$ when $x = 0$ (b) $y = \dfrac{\sqrt{x}(2 - x^2)}{x}$ when $x = 4$

**Problem 17** Find all points on the curve $y = \frac{x^5}{5} - x + 1$ where the tangent line is horizontal.

**Exercise 5** Problems 11.2: even-numbered problems 2-74; 82, 85, 88 ■

# Unit 6

## The Derivative as a Rate of Change

1. If $y = f(x)$, then average rate of change of $y$ with respect to $x$ over the interval from $x$ to $x + \Delta x$ is
$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x},$$
and instantaneous rate of change of $y$ with respect to $x$ is
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

2. **Applications of Rate of Change to Economics**

   (a) A manufacturer's **total-cost function**, $c = f(q)$, gives the total cost $c$ of producing and marketing $q$ units of a product.

   (b) The rate of change of $c$ with respect to $q$ is called the **marginal cost**. Thus,
$$\text{marginal cost } = \frac{dc}{dq}.$$

   (c) We interpret the marginal cost as the approximate cost of one additional unit of output.

   (d) If $c$ is a total cost of producing $q$ units of a product, then the **average cost per unit**, $\overline{c}$, is $\overline{c} = \dfrac{c}{q}$.

   (e) A manufacturer's **total-revenue function**, $r = f(q)$, gives the total monetary value received for selling $q$ units.

   (f) If $p$ denotes the demand function giving the price per unit for selling $q$ units, then the total revenue $r$ is given by $r = pq$.

   (g) The rate of change of $r$ with respect to $q$ is called the **marginal revenue**. Thus,
$$\text{marginal revenue } = \frac{dr}{dq}.$$

   (h) We interpret the marginal revenue as the approximate revenue received from selling one additional unit of output.

3. The **relative rate of change** of $f(x)$ is $\dfrac{f'(x)}{f(x)}$.

4. The **percentage rate of change** of $f(x)$ is $\dfrac{f'(x)}{f(x)} \cdot 100\%$.

**Problem 18** For the cost function $c = 0.3q^2 + 3.5q + 9$, what is the marginal cost when $q = 10$? Interpret the result.

**Problem 19** For a certain manufacturer, the revenue obtained from the sale of $q$ units of a product is given by $r = 10q - 0.1q^2$.

(a) How fast does $r$ change with respect to $q$?

(b) When $q = 25$, find the marginal revenue and interpret the result.

**Problem 20** A manufacturer of mountain bikes has found that when 20 bikes are produced per day, the average cost is \$150 and the marginal cost is \$125. Based on that information, approximate the total cost of producing 21 bikes per day.

**Exercise 6** Problems 11.3: 6, 9, 16, 22, 26, 40, 42

# Unit 7

# More Differentiation Rules

1. **The Product Rule**

   If $f$ and $g$ are differentiable functions, then the product $fg$ is differentiable, and

   $$\frac{d}{dx}\big(f(x)g(x)\big) = f(x)g'(x) + g(x)f'(x).$$

   That is,

   $$\frac{d}{dx}(\text{Hi} \cdot \text{Ho}) = \text{Hi di Ho} + \text{Ho di Hi}.$$

2. **The Quotient Rule**

   If $f$ and $g$ are diffentiable functions and $g(x) \neq 0$, then the quotient $f/g$ is also differentiable, and

   $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{\big(g(x)\big)^2}.$$

   That is,

   $$\frac{d}{dx}\left(\frac{\text{Hi}}{\text{Ho}}\right) = \frac{\text{Ho di Hi} - \text{Hi di Ho}}{\text{Ho Ho}}.$$

3. **The Chain Rule**

   If $y$ is a differentiable function of $u$ and $u$ is a differentiable function of $x$, then $y$ is a differentiable function of $x$ and

   $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

4. **The Power Rule**

   If $u$ is a differentiable function of $x$ and $n$ is any real number, then

   $$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

**Problem 21** Differentiate the functions.

(a) $y = (\sqrt{x} + 5x - 2)\left(\sqrt[3]{x} - \frac{3}{x}\right)$

(b) $y = \dfrac{8x^2 - 2x + 1}{2x^2 - 3x + 2}$

(c) $y = 3 - 12x^3 + \dfrac{1 - \frac{5}{x^2+2}}{x^2 + 5}$

(d) $y = (2x^3 - 8x)^{-12}$

(e) $y = \dfrac{(3x+2)^5}{(4x-5)^2}$

(f) $y = \sqrt[3]{(x-2)^2(x+2)}$

**Problem 22** The cost $c$ of producing $q$ units of a product is given by

$$c = 5500 + 12q + 0.2q^2.$$

If the price per unit $p$ is given by the equation

$$q = 900 - 1.5p,$$

then find the rate of change of cost with respect to price per unit when $p = 85$.

**Problem 23** Suppose $y = f(x)$, where $x = g(t)$. Given that $g(2) = 3$, $g'(2) = 4$, $f(2) = 5$, $f'(2) = 6$, $g(3) = 7$, $g'(3) = 8$, $f(3) = 9$, and $f'(3) = 10$, determine the value of

$$\left.\frac{dy}{dt}\right|_{t=2}.$$

**Exercise 7** Problems 11.4: even-numbered problems 24-46, 54, 56

Problems 11.5: 2, 4, 6, 8, 10, 22, 32, 40, 52, 68 ■

# Unit 8

# Derivatives of Logarithmic and Exponential Functions

1. **Derivatives of Logarithmic Functions**

(a) $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$ for $x \neq 0$.

(b) $\dfrac{d}{dx}\ln|u| = \dfrac{1}{u}\cdot\dfrac{du}{dx}$ for $u \neq 0$.

(c) $\dfrac{d}{dx}\log_b|u| = \dfrac{1}{(\ln b)u}\cdot\dfrac{du}{dx}$ for $u \neq 0$.

**Problem 24** Differentiate the functions.

(a) $y = \ln\left(\dfrac{2x+3}{3x-4}\right)$ (b) $y = \ln^2(2x+11)$

(c) $y = \ln(x^3\sqrt[4]{2x+1})$ (d) $y = \ln(x+\sqrt{1+x^2})$

**Problem 25** Find an equation of the tangent line to the curve $y = x(\ln x - 1)$ at the point where $x = e$.

**Problem 26** A manufacturer's average-cost function, in dollars, is given by

$$\bar{c} = \frac{500}{\ln(q+20)}.$$

Find the marginal cost (rounded to two decimal places) when $q = 50$.

2. **Derivatives of Exponential Functions**

(a) $\dfrac{d}{dx}e^x = e^x$.

(b) $\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}$.

(c) $\dfrac{d}{dx}b^u = b^u(\ln b)\dfrac{du}{dx}$.

**Problem 27** Differentiate the functions.

(a) $y = e^{2x}(x+6)$ (b) $y = 2^{1+\sqrt{x}}$

(c) $y = \dfrac{e^x-1}{e^x+1}$ (d) $y = e^{x^2\ln x^2}$

**Problem 28** If $f(x) = 5^{x^2\ln x}$, find $f'(1)$.

**Problem 29** If $w = e^{x^3-4x} + x\ln(x-1)$ and $x = \dfrac{t+1}{t-1}$, find $\dfrac{dw}{dt}$ when $t = 3$.

**Problem 30** Calculate the relative rate of change of

$$f(x) = 10^{-x} + \ln(8+x) + 0.01e^{x-2}$$

when $x = 2$. Round your answer to four decimal places.

**Exercise 8** Problems 12.1: even-numbered problems 2-44, 46, 48
Problems 12.2: even-numbered problems 2-28, 38, 42 ■

# Unit 9

# Implicit Differentiation, Higher-Order Derivatives

1. Implicit Differentiation Procedure

   For an equation that we assume defines $y$ implicitly as a differentiable function of $x$, the derivative $\dfrac{dy}{dx}$ can be found as follows:

   (a) Differentiate both sides of the equation with respect to $x$.

   (b) Collect all terms involving $\dfrac{dy}{dx}$ on one side of the equation, and collect all other terms on the other side.

   (c) Solve for $\dfrac{dy}{dx}$.

2. If we differentiate $f'(x)$, the resulting function $(f')'(x)$ is called the **second derivative** of $f$ at $x$. It is denoted

$$f''(x) \quad \text{or} \quad \frac{d^2}{dx^2}(f(x)),$$

   which is read "$f$ double prime of $x$."

   Continuing in this way, we get **higher-order derivatives**.

**Problem 31** Find $\dfrac{dy}{dx}$ by implicit differentiation.

(a) $x^2 + xy - 2y^2 = 0$ (c) $x = \sqrt{y} + \sqrt[4]{y}$ (c) $(1 + e^{3xy})^2 = 3 + \ln(x - y^2)$

**Problem 32** Find an equation of the tangent line to the curve $x^3 + xy + y^2 = -1$ at the point $(-1, 1)$.

**Problem 33** Find the rate of change of $q$ with respect to $p$ for

$$p = \frac{20}{(q+5)^2}.$$

**Problem 34** If $f(x) = 6x^3 - 12x^2 + 6x - 2$, find all higher-order derivatives.

**Problem 35** If $y = e^{x^2}$, find $\dfrac{d^2y}{dx^2}$.

**Problem 36** If $y = \dfrac{16}{x+4}$, find $\dfrac{d^2y}{dx^2}$ and evaluate it when $x = 4$.

**Problem 37** If $f(x) = x \ln x$, find the rate of change of $f''(x)$.

**Problem 38** Show that the equation

$$f''(x) + 4f'(x) + 4f(x) = 0$$

is satisfied if $f(x) = (3x - 5)e^{-2x}$.

**Problem 39** If $c = 0.2q^2 + 2q + 500$ is a cost function, how fast is marginal cost function changing when $q = 97.357$?

**Exercise 9** Problems 12.4: 5, 18, 26, 28, 35

Problems 12.7: 2, 7, 16, 20, 38 ■

# Unit 10

# Partial Derivatives

**Definition 5** If $z = f(x, y)$, the **partial derivative of** $f$ **with respect to** $x$, denoted $f_x$ or $\dfrac{\partial f}{\partial x}$, is the function of two variables, given by

$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

provided that the limit exists.

the **partial derivative of** $f$ **with respect to** $y$, denoted $f_y$ or $\dfrac{\partial f}{\partial y}$, is the function of two variables, given by

$$f_y(x, y) = \frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

provided that the limit exists.

## 2. Procedure to Find $f_x(x, y)$ and $f_y(x, y)$

To find $f_x$, treat $y$ as constants, and differentiate $f$ with respect to $x$ in the usual way.

To find $f_y$, treat $x$ as constants, and differentiate $f$ with respect to $y$ in the usual way.

**Problem 40** If $f(x, y) = xy^2 + x^2y$, find $f_x(x, y)$ and $f_y(x, y)$. Also, find $f_x(3, 4)$ and $f_y(3, 4)$.

**Problem 41** (a) If $z = 3x^3y^3 - 9x^2y + xy^2 + 4y$, find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$, $\left.\dfrac{\partial z}{\partial x}\right|_{(1,0)}$, and $\left.\dfrac{\partial z}{\partial y}\right|_{(1,0)}$.

(b) If $w = x^2e^{2x+3y}$, find $\partial w/\partial x$ and $\partial w/\partial y$.

**Problem 42** If $p = g(r, s, t, u) = \dfrac{rsu}{rt^2 + s^2t}$, find $\dfrac{\partial p}{\partial s}$, $\dfrac{\partial p}{\partial t}$, and $\left.\dfrac{\partial p}{\partial t}\right|_{(0,1,1,1)}$.

## 3. Applications of Partial Derivatives

If $z = f(x, y)$, then $\dfrac{\partial z}{\partial x}$ is the rate of change of $z$ with respect to $x$ when $y$ is held fixed.

Similarly, $\dfrac{\partial z}{\partial y}$ is the rate of change of $z$ with respect to $y$ when $x$ is held fixed.

**Problem 43** A company manufacturers two types of skis, the Lightning and the Alpine models. Suppose the joint-cost function for producing $x$ pairs of the Lightning model and $y$ pairs of the Alpine model per week is

$$c = 0.07x^2 + 75x + 85y + 6000,$$

where $c$ is expressed in dollars. Determine the marginal costs $\partial c/\partial x$ and $\partial c/\partial y$ when $x = 100$ and $y = 50$, and interpret the results.

4. If the function $P = f(\ell, k)$ gives the output $P$ when the producer uses $\ell$ units of labor and $k$ units of capital, then this function is called a **production function**. We define the **marginal productivity with respect to** $\ell$ to be $\partial P/\partial \ell$. Likewise, the **marginal productivity with respect to** $k$ is $\partial P/\partial k$.

**Problem 44** A manufacturer of a popular toy has determined that the production function is $P = \sqrt{\ell k}$, where $\ell$ is the number of labor-hours per week and $k$ is the capital (expressed in hundreds of dollars per week) required for a weekly production of $P$ gross of the toy. (One gross is 144 units.) Determine the marginal productivity functions, and evaluate them when $\ell = 400$ and $k = 16$. Interpret the results.

**Exercise 10** Problems 17.1: 5, 6, 10, 17, 28, 35

Problems 17.2: 2, 3, 4, 5, 10, 12 ∎

# Unit 11

# Extrema

1. A function $f$ is said to be **increasing** on an interval $I$ when, for any two numbers $x_1, x_2$ in $I$, if $x_1 < x_2$, then $f(x_1) < f(x_2)$. A function $f$ is **decreasing** on an interval $I$ when, for any two numbers $x_1, x_2$ in $I$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

2. **Criteria for Increasing and Decreasing Function**

   Let $f$ be differentiable on the interval $(a, b)$. If $f'(x) > 0$ for all $x$ in $(a, b)$, then $f$ is increasing on $(a, b)$. If $f'(x) < 0$ for all $x$ in $(a, b)$, then $f$ is decreasing on $(a, b)$.

3. A function $f$ has a **relative maximum** at $a$ if there is an open interval containing $a$ on which $f(a) \geq f(x)$ for all $x$ in the interval. The relative maximum value is $f(a)$.

   A function $f$ has a **relative minimum** at $a$ if there is an open interval containing $a$ on which $f(a) \leq f(x)$ for all $x$ in the interval. The relative minimum value is $f(a)$.

4. A function $f$ has an **absolute maximum** at $a$ if $f(a) \geq f(x)$ for all $x$ in the domain of $f$. The absolute maximum value is $f(a)$.

   A function $f$ has an **absolute minimum** at $a$ if $f(a) \leq f(x)$ for all $x$ in the domain of $f$. The absolute minimum value is $f(a)$.

5. **A Necessary Condition for Relative Extrema**

   If $f$ has a relative extremum at $a$, then $f'(a) = 0$ or $f'(a)$ does not exist.

6. For $a$ in the domain of $f$, if either $f'(a) = 0$ or $f'(a)$ does not exist, then $a$ is called a **critical value** for $f$. If $a$ is a critical value, then the point $\big(a, f(a)\big)$ is called a **critical point** for $f$.

7. **Criteria for Relative Extrema**

   Suppose $f$ is continuous on an open interval $I$ that contains the critical value $a$ and $f$ is differentiable on $I$, except possibly at $a$.

   (a) If $f'(x)$ changes from positive to negative as $x$ increases through $a$, then $f$ has a relative maximum at $a$.

   (b) If $f'(x)$ changes from negative to positive as $x$ increases through $a$, then $f$ has a relative minimum at $a$.

**Problem 45** Determine where the function is increasing or decreasing, and determine where relative maxima and minima occur.

(a) $y = x^2 + 4x + 3$

(b) $y = x^3 - \dfrac{5}{2}x^2 - 2x + 6$

(c) $y = 3x^5 - 5x^3$

(d) $y = \sqrt[3]{x}(x - 2)$

(e) $y = \dfrac{x^2}{2-x}$

(f) $y = \sqrt[3]{x^3 - 9x}$

8. **Extreme-Value Theorem**

**Theorem 3** If a function is continuous on a closed interval, then the function has **both** a maximum value **and** a minimum value on that interval.

9. **Procedure to Find Absolute Extrema for a Function $f$ That is Continuous on $[a, b]$.**

**Step 1**. Find the critical values of $f$.

**Step 2**. Evaluate $f(x)$ at the endpoints $a$ and $b$ and at the critical values in $(a, b)$.

**Step 3**. The maximum value of $f$ is the greatest of the values found in step 2. The minimum value of $f$ is the least of the values found in step 2.

**Problem 46** Find the absolute extrema of the given function on the given interval.

(a) $f(x) = -2x^2 - 6x + 5, \quad [-3, 2]$

(b) $f(x) = \dfrac{1}{4}x^4 - \dfrac{3}{2}x^2, \quad [0, 1]$

(c) $f(x) = 3x^4 - x^6, \quad [-1, 2]$

(d) $f(x) = \dfrac{x}{x^2+1}, \quad [0, 2]$

**Exercise 11** Problems 13.1: 2, 10, 14, 18, 20, 36, 54, 58

Problems 13.2: 1, 3, 5, 7, 11 ■

# Unit 12

## Concavity
## Applied Maxima and Minima

1. Let $f$ be differentiable on the interval $(a, b)$. Then $f$ is said to be **concave up [concave down]** on $(a, b)$ if $f'$ is increasing [decreasing] on $(a, b)$.

2. **Criteria for Concavity**

   Let $f'$ be differentiable on the interval $(a, b)$. If $f''(x) > 0$ for all $x$ in $(a, b)$, then $f$ is concave up on $(a, b)$. If $f''(x) < 0$ for all $x$ in $(a, b)$, then $f$ is concave down on $(a, b)$.

3. A function $f$ has an **inflection point** at $a$ if and only if $f$ is continuous at $a$ and $f$ changes concavity at $a$.

4. **Second-Derivative Test for Relative Extrema**

   **Theorem 4** Suppose $f'(a) = 0$.

   If $f''(a) < 0$, then $f$ has a relative maximum at $a$.

   If $f''(a) > 0$, then $f$ has a relative minimum at $a$.

   **Problem 47** Sketch the curve.

   (a) $y = x^3 - 9x^2 + 24x - 19$

   (b) $y = x^3 - 25x^2$

   (c) $y = 3x^4 - 4x^3 + 1$

   (d) $y = x^2(x - 1)^2$

5. **Guide for Solving Applied Max-Min Problems**

Step 1. When appropriate, draw a diagram that reflects the information in the problem.

Step 2. Set up a function for the quantity that you want to maximize or minimize.

Step 3. Express the function in step 2 as a function of one variable only, and note the domain of this function.

Step 4. Find the critical values of the function. After testing each critical value, determine which one gives the absolute extreme value you are seeking. If the domain of the function includes endpoints, be sure to also examine function values at these endpoints.

Step 5. Based on the results of step 4, answer the question(s) posed in the problem.

**Problem 48** Find two nonnegative numbers whose sum is 20 and for which the product of twice one number and the square of the other number will be a maximum.

**Problem 49** The demand equation for a monopolist's product is $p = -5q + 30$. At what price will revenue be maximized.

**Problem 50** For a monopollist's product, the demand function is $p = \dfrac{40}{\sqrt{q}}$ and the average-cost function is $\overline{c} = \dfrac{1}{3} + \dfrac{2000}{q}$. Find the profit-maximizing price and output. At this level, show that marginal revenue is equal to marginal cost.

**Problem 51** A container manufacturer is designing a rectangular box, open at the top and with a square base, that is to have a volume of 32 $\text{ft}^3$. If the box is to require the least amount of material, what must be its dimensions?

**Exercise 12** Problems 13.3: 41–50

Problems 13.6: 1, 4, 11, 12, 13 ■

# Unit 13
# Integration

1. An **antiderivative** of a function $f$ is a function $F$ such that
$$F'(x) = f(x).$$
2. Any two antiderivatives of a function differ only by a constant.
3. The **indefinite integral** of any function $f$ with respect to $x$ is written $\int f(x)\,dx$ and denotes the most general antiderivative of $f$. Hence,
$$\int f(x)\,dx = F(x) + C \quad \text{if and only if} \quad F'(x) = f(x).$$
4. **Basic Integration Formulas**
   (a) $\int k\,dx = kx + C, \quad k \text{ is a constant}$
   (b) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
   (c) $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \int \frac{dx}{x} = \ln|x| + C \quad \text{for } x \neq 0$
   (d) $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C \quad \text{for } a \neq 0$
   (e) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C \quad \text{for } a \neq 0$
   (f) $\int kf(x)\,dx = k\int f(x)\,dx, \quad k \text{ is a constant}$
   (g) $\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
5. **Techniques of Integration**
   (a) Guessing a likely antiderivative
   (b) Substitution

**Problem 52** Determine the indefinite integrals.

(a) $\int \left(\sqrt[3]{x} - \frac{3}{\sqrt[3]{x}}\right) dx$

(b) $\int \big(e^{2x} - x^3(\sqrt{x}+1)\big)\,dx$

(c) $\int \frac{(x^3+1)^2}{x^2}\,dx$

(d) $\int \frac{2x^2}{3-4x^3}\,dx$

(e) $\displaystyle\int \sqrt{\sqrt{x}+1}\,dx$

(f) $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

(g) $\displaystyle\int (1-e^x)^5 e^{2x}\,dx$

**Problem 53** A manufacturer has determined that the marginal-cost function is

$$\frac{dc}{dq} = 0.003q^2 - 0.4q + 40$$

where $q$ is the number of units produced. If marginal cost is \$27.50 when $q = 50$ and fixed costs are \$5000, what is the average cost of producing 100 units?

**Problem 54** The marginal-cost function for a manufacturer's product is given by

$$\frac{dc}{dq} = \frac{9}{10}\sqrt{q}\sqrt{0.04q^{3/2}+4}$$

where $c$ is the total cost in dollars when $q$ units are produced. Fixed costs are \$360.

(a) Determine the marginal cost when 25 units are produced.

(b) Find the total cost of producing 25 units.

**Exercise 13** Problems 14.2: 19, 37, 46, 52

Problems 14.3: 3, 6, 15

Problems 14.4: 43, 51, 74, 75

Problems 14.5: 21, 25, 40, 42, 52 ■

# Unit 14
# Integration (continued)

1. **Fundamental Theorem of Integral Calculus**

   If $f$ is continuous on the interval $[a, b]$ and $F$ is any antiderivative of $f$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

2. **Properties of the Definite Integral**

   (a) If $f$ is continuous and $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx$ can be interpreted as the area of the region bounded by the curve $y = f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$.

   (b) $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ where $k$ is a constant

   (c) $\displaystyle\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

   (d) $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

   The variable of integration is a **dummy variable** in the sense that any other variable produces the same result–that is, the same number.

   (e) If $f$ is continuous on an interval $I$ and $a, b$, and $c$ are in $I$, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

   (f) We define:

   i. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ if $a > b$

   ii. $\displaystyle\int_a^a f(x)\,dx = 0$

**Problem 55** Determine the definite integrals.

(a) $\displaystyle\int_1^8 (x^{1/3} - x^{-1/3})\,dx$

(b) $\displaystyle\int_{-1/3}^{20/3} \sqrt{3x+5}\,dx$

(c) $\displaystyle\int_0^1 \frac{2x^3 + x}{x^2 + x^4 + 1}\,dx$

(d) $\displaystyle\int_0^3 \frac{3x}{\sqrt{4-x}}\,dx$

(e) $\displaystyle\int_3^{27} 3(\sqrt{3x} - 2x + 1)\,dx$

(f) $\int_1^2 5x\sqrt{5-x^2}\,dx$

**Problem 56** Find the area of the region bounded by the graphs of the given equation.

(a) $y = 2x - 1$, $y = x^2 - 4$ and the vertical lines $x = 1$ and $x = 2$

(b) $y = (x-1)^2, \quad y = x - 1$

(c) $y = 10 - x^2, \quad y = 4$

(d) $y = x^2, \quad y = \sqrt{x}$

(e) $y = x^3 - 1, \quad y = x - 1$

**Problem 57** A manufacturer's marginal-cost function is $\dfrac{dc}{dq} = 0.6q + 2$. If production is presently set at $q = 80$ units per week, how much more would it cost to increase production to 100 units per week?

**Problem 58** The demand function for a product is $p = 100 - 0.05q$ where $p$ is the price per unit (in dollars) for $q$ units. The supply function is $p = 10 + 0.1q$. Determine consumers' surplus and producers' surplus under market equilibrium.

**Problem 59** The demand equation for a product is $p = (q-4)^2$ and the supply equation is $p = q^2 + q + 7$ where $p$ (in thousands of dollars) is the price per 100 units when $q$ hundred units are demanded or supplied. Determine consumers' surplus under market equilibrium.

**Exercise 14** Problems 14.7: 12, 30, 39

Problems 14.9: 49, 51, 55

Problems 14.10: 3, 4 ■