Department of Mathematics
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Introduction to Arithmetic Dynamics

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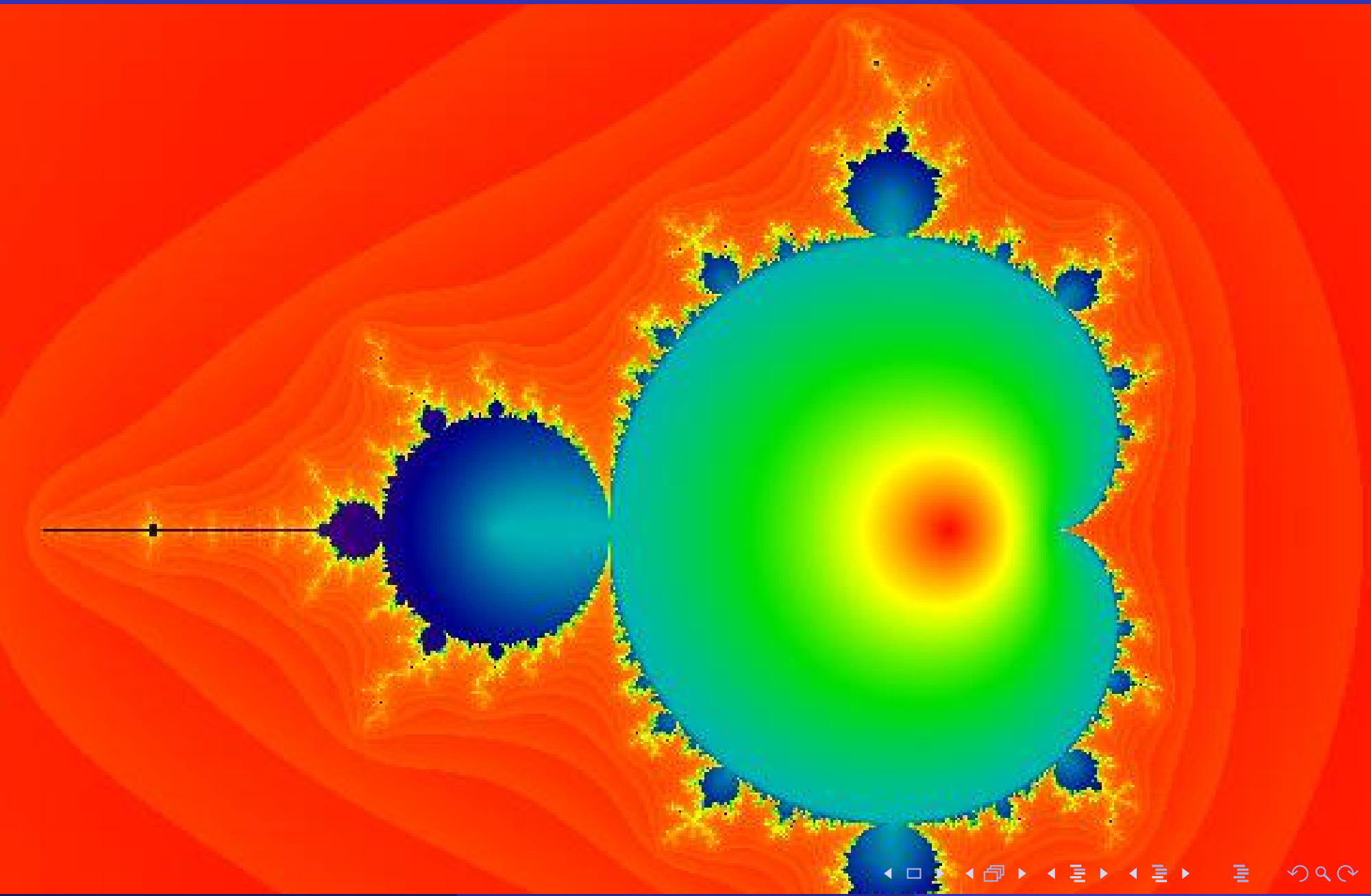
Mandelbrot set $= \{c \in \mathbb{C} | z_0 = 0, z_{n+1} = z_n^2 + c \text{ is bounded}\}$



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- *The (forward) **orbit** of $z \in S$ by f is defined as*

$$\mathscr{O}_f(z) := \{z, f(z), f^{(2)}(z), f^{(3)}(z), \dots\}.$$

Some Definitions (continued)

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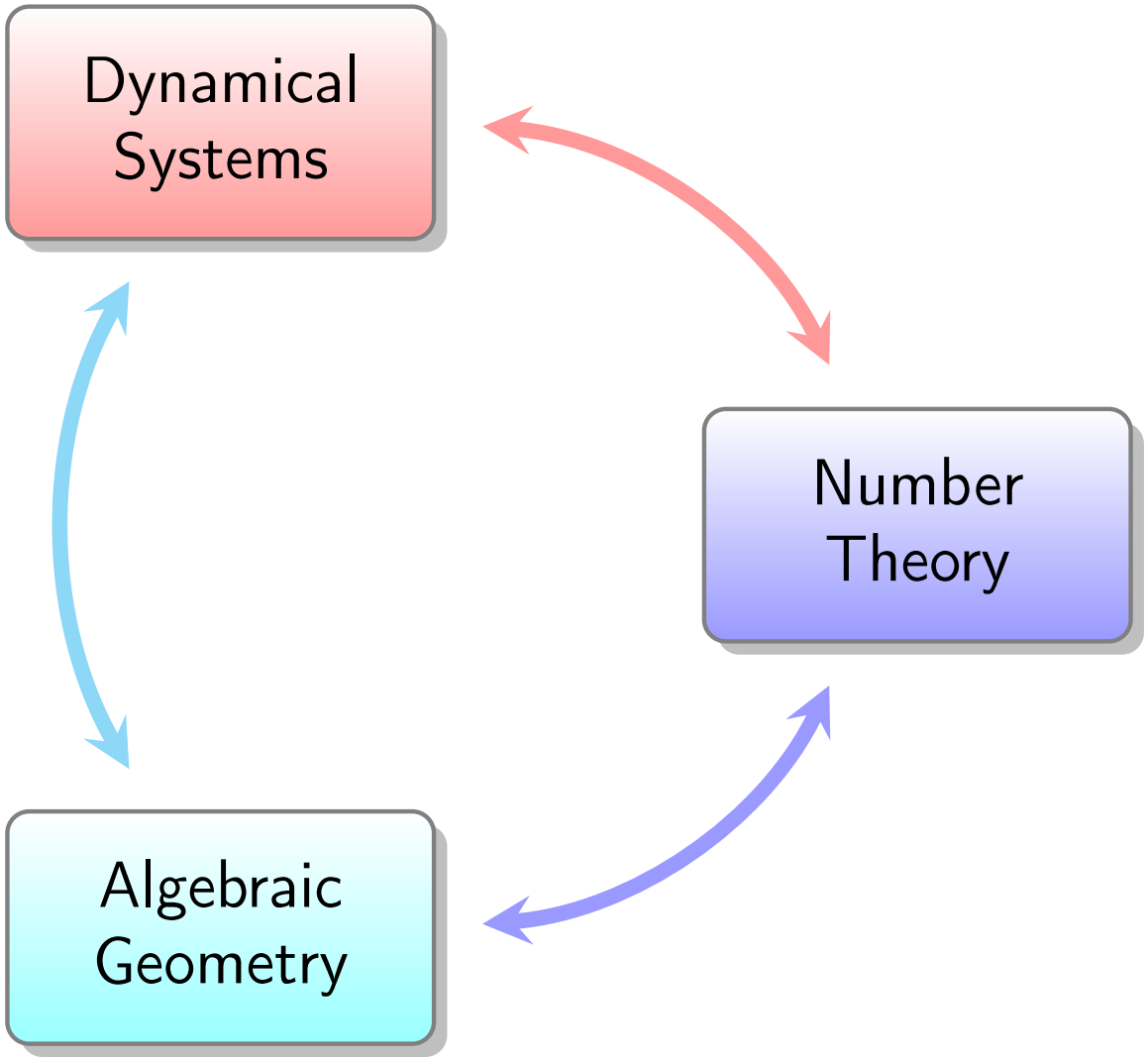
- *We say that a point $z \in S$ is **wandering** if z is not preperiodic.*

For arithmetic interests, we can let S be $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{K}, \ldots$

Arithmetic Dynamics



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No, it's not preperiodic.
Is $z = \frac{1}{4}$ preperiodic?

$$\frac{1}{4} \mapsto -\frac{7}{4} \mapsto -\frac{5}{4} \mapsto -\frac{1}{4} \mapsto -\frac{7}{4}$$

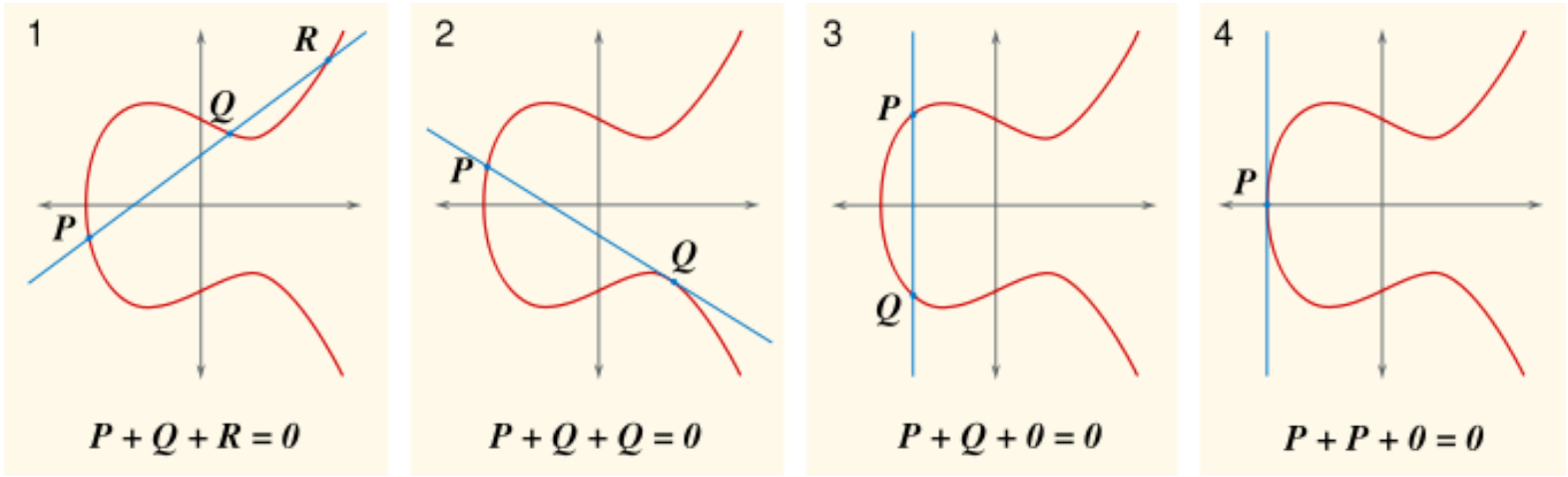
Dictionary

TABLE 1. An arithmetical/dynamical dictionary [218, §6.5]

Arithmetic Geometry		**Dynamical Systems**
rational and integral points on varieties	⟷	rational and integral points in orbits
torsion points on abelian varieties	⟷	periodic and preperiodic points of rational maps
abelian varieties with complex multiplication	⟷	post-critically finite rational maps

Figure: From "Current Trends and Open Problems in Arithmetic Dynamics"

Rational map of $E(\mathbb{C}) : y^2 = x^3 + ax + b$



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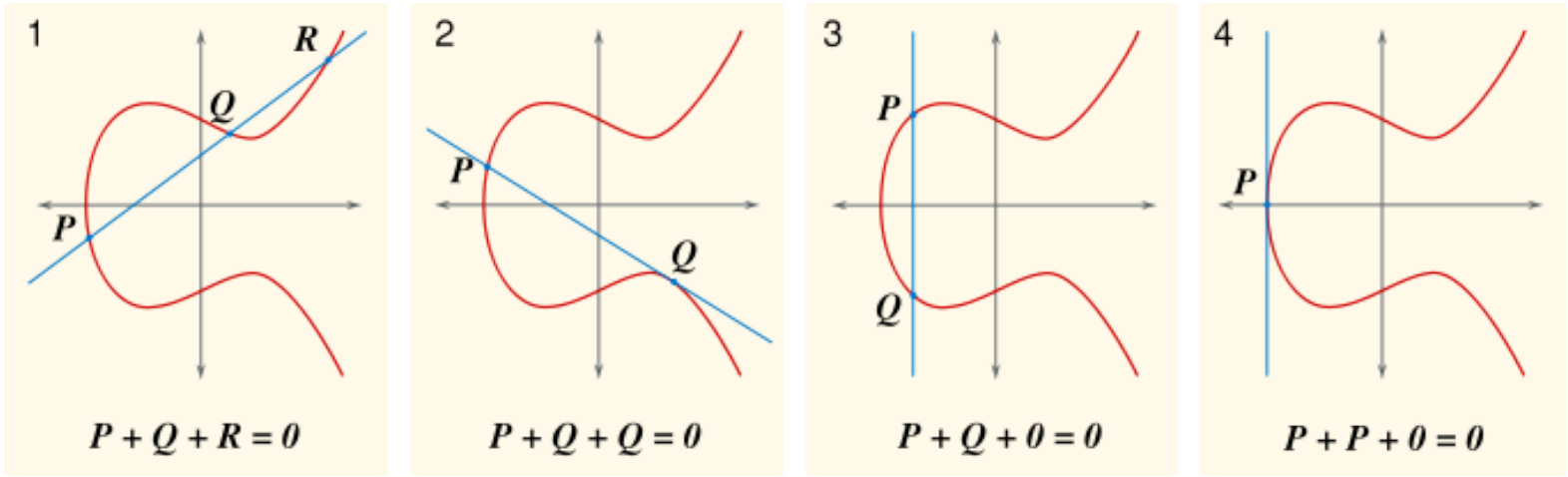

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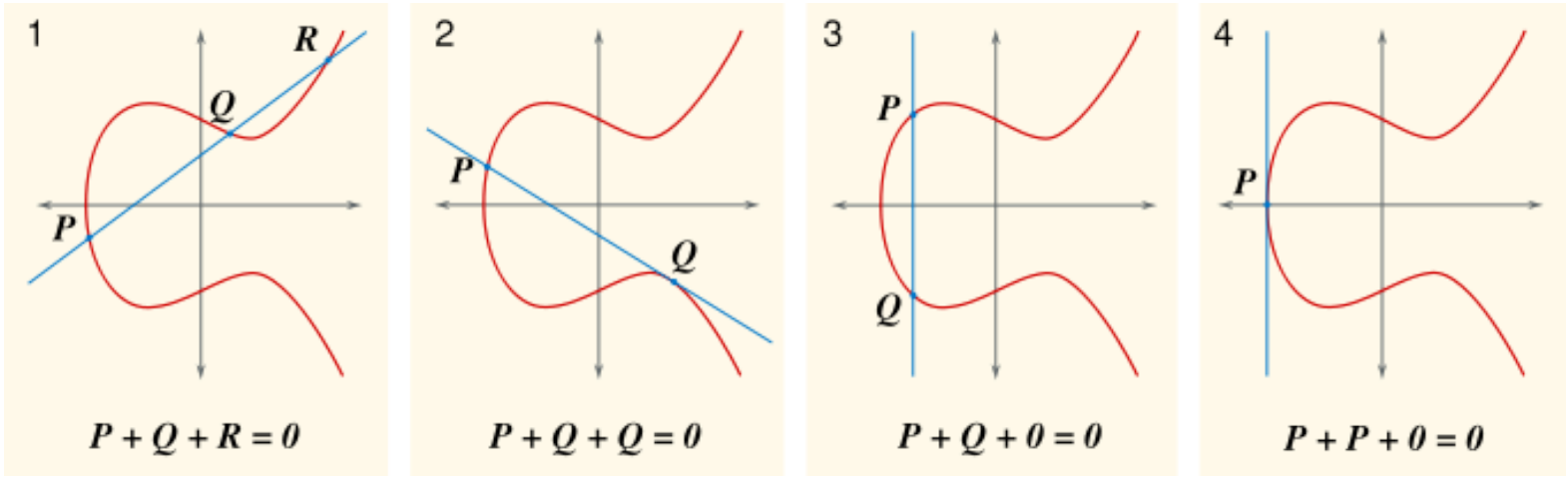

$$E(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C}), (x, y) \longmapsto x \qquad \begin{array}{ccc} E(\mathbb{C}) & \xrightarrow{[2]} & E(\mathbb{C}) \\ \downarrow x & & \downarrow x \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{\phi_{E,2}} & \mathbb{P}^1(\mathbb{C}) \end{array}$$

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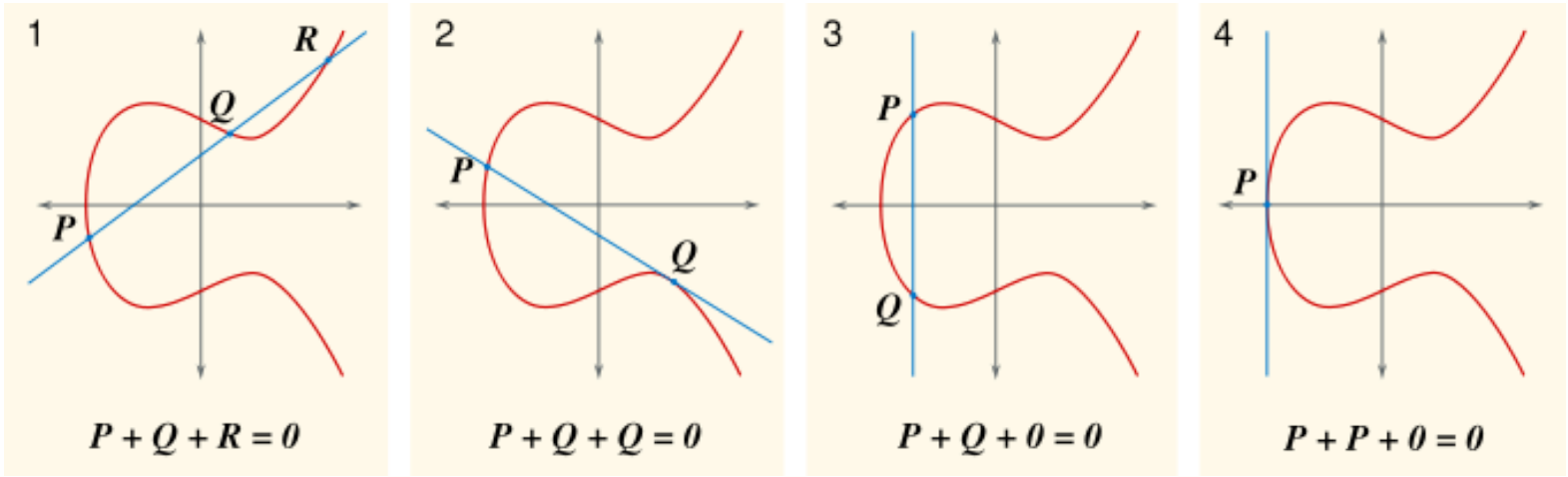

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$$\phi_{E,2}(x) = \frac{x^4 - 2ax^2 - 8bx + a^2}{4x^3 + 4ax + 4b} \text{ (Lattès map)}$$

Torsion points of $E(\mathbb{C}) : y^2 = x^3 + ax + b$

Definition

Let $E(\mathbb{C})$ be an elliptic curve. $(x, y) \in E(\mathbb{C})$ is a **torsion point** if (x, y) has finite order under addition.

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Remark The set of torsion points of an elliptic curve forms a group called "torsion subgroup".

Some results on torsion subgroups

- **Mordell-Weil Theorem(1922)** $E(\mathbb{Q}) \simeq T \oplus \mathbb{Z}^r$, where $T = E_{tors}(\mathbb{Q})$ is a finite group.

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- **Mazur Theorem(1978)** T is isomorphic to one of the following:
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  $\mathbb{Z}_n$ with $1 \leq 10$ or $n = 12$,
  $\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}$ with $1 \leq n \leq 4$.
- **Merel Theorem(1998)** For every integer $d \geq 1$ there is a constant $N(d)$ such that for all number fields $K/\mathbb{Q}$ of degree at most d and all elliptic curves $E(K)$,

$$E_{tors}(K) \leq N(d).$$

Conjecture (Morton-Silverman Uniform Boundedness, 1994)

There exists a bound $B = B(D, N, d)$ such that if K is a number field of degree D, and $\phi : \mathbb{P}^N(K) \longrightarrow \mathbb{P}^N(K)$ is a morphism of degree $d \geq 2$ defined over K, then the number of preperiodic points of ϕ is bounded by B.

This conjecture is remarkably strong. For $(D, N, d) = (1, 1, 4)$, the conjecture implies that the size of the torsion subgroup of an elliptic curve is uniformly bounded. This can be done via the associated Lattès map. Similarly, for $(D, N, d) = (D, 1, 4)$, the conjecture implies the Merel's theorem.

Theorem (Northcott, 1950)

Let $\phi \in K(z)$ of degree $d \geq 2$. Then

$$\mathrm{PrePer}(\phi, K) < \infty.$$

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Conjecture (Morton-Silverman, $\mathbb{Q}$ version)

For any integer $d \geq 2$, there is a constant $C(d)$ such that for any $\phi \in \mathbb{Q}(z)$ of degree d,

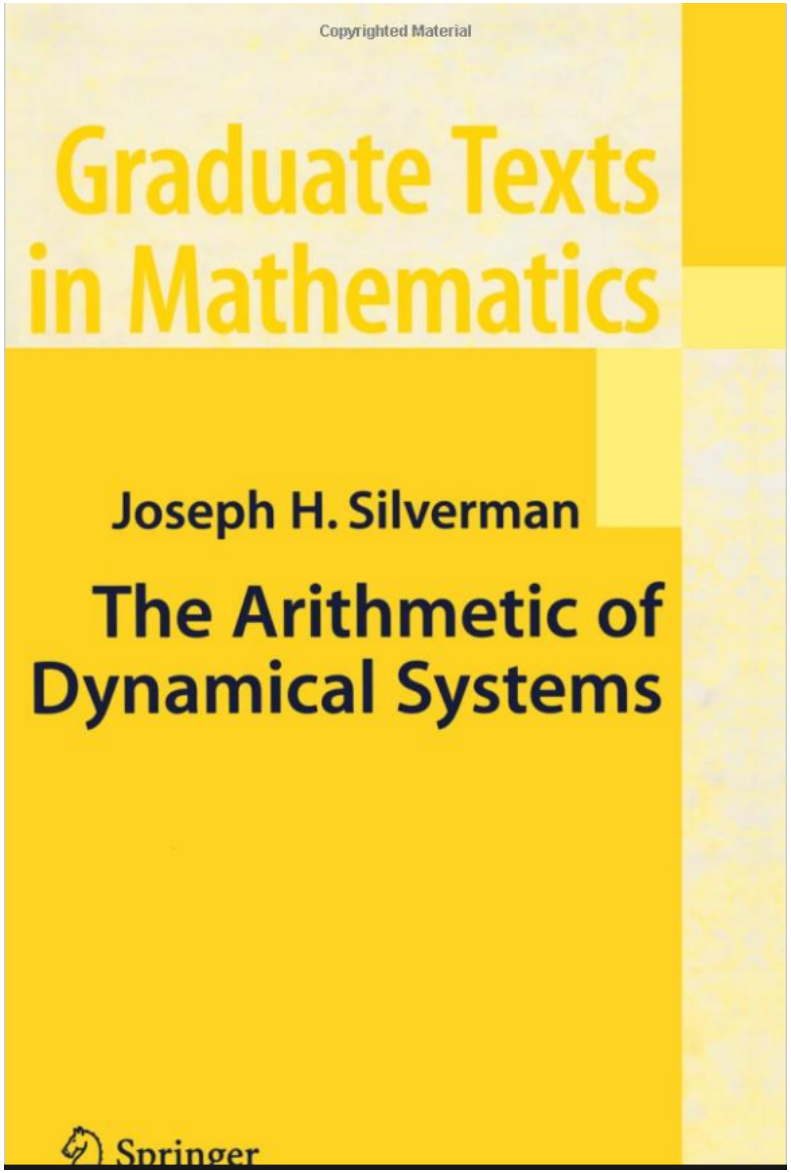
$$\mathrm{PrePer}(\phi) \leq C(d).$$

Theorem (Doyle-Poonen, 2020)

Fix integers $d \geq 2$, $D \geq 1$, and $N \geq 1$. Then there exists $B = B(d, D, N) > 0$ such that for every number field K satisfying $[K : Q] \leq D$ and every $c \in K$, the number of preperiodic points of $z^d + c$ in K with eventual period at most N is at most B.

In the paper, Doyle and Poonen also prove the **strong uniform boundedness theorem for preperiodic points over function fields**.

The ADS by Joseph H. Silverman



CURRENT TRENDS AND OPEN PROBLEMS IN ARITHMETIC DYNAMICS, 2019

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CURRENT TRENDS AND OPEN PROBLEMS IN ARITHMETIC DYNAMICS

ROBERT BENEDETTO, PATRICK INGRAM, RAFE JONES, MICHELLE MANES, JOSEPH H. SILVERMAN, AND THOMAS J. TUCKER

ABSTRACT. Arithmetic dynamics is the study of number theoretic properties of dynamical systems. A relatively new field, it draws inspiration partly from dynamical analogues of theorems and conjectures in classical arithmetic geometry and partly from p-adic analogues of theorems and conjectures in classical complex dynamics. In this article we survey some of the motivating problems and some of the recent progress in the field of arithmetic dynamics.

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THANK YOU!