Three Ways to Determine a Determinant

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Outline

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Abstract

A conjecture or a problem in mathematics acts like a muse that inspires further developments, new concepts, and a rich tapestry of fundamental ideas.

M. Ram Murty – Problems in the Theory of Modular Forms

We discuss three different methods to determine the value of a determinant whose entries are powers of the Fibonacci numbers. One of the methods has led us to rediscover linear homogeneous recurrence relations satisfied by the powers of the Fibonacci numbers.

Outline

Fibonacci Numbers

- The sequence $\{F_n\}$ of Fibonacci numbers (OEIS: A000045) is defined by

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}, \quad n \geq 2.$$

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- The first few terms of Fibonacci numbers are:

$$\underline{0}, \underline{1}, \underline{1}, 2, 3, 5, \underline{8}, 13, 21, 34, 55, 89, \underline{144}, \ldots,$$

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- The first few terms of Fibonacci numbers are:

$$\underline{0},\ \underline{1},\ \underline{1},\ 2,\ 3,\ 5,\ \underline{8},\ 13,\ 21,\ 34,\ 55,\ 89, \underline{144},\ \ldots,$$

- Some interesting results:
  1. Fibonacci Perfect Powers Theorem (Bugeaud et al, 2003): The only perfect Fibonacci powers are $0, 1, 8$, and 144.

1. F. Buchanan, "Nth Powers in the Fibonacci Series," Am. Math. Monthly, 71 (1964), 647-649.
2. F. Buchanan, Retraction of "Nth Powers in the Fibonacci Series," Am. Math. Monthly, 71 (1964), 1112.

Fibonacci Numbers (cont.)

2. If $a \geq c, d$ and $a + b = c + d$, then

$$F_a F_b - F_c F_d = (-1)^{b+1} F_{a-c} F_{a-d}.$$

Fibonacci Numbers (cont.)

2. If $a \geq c, d$ and $a + b = c + d$, then

$$F_a F_b - F_c F_d = (-1)^{b+1} F_{a-c} F_{a-d}.$$

- With $a = n + m$, $b = 1$, $c = n + 1$, $d = m$, we obtain

$$F_{n+m} = F_n F_{m-1} + F_{n+1} F_m$$

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- With $a = n + m$, $b = 1$, $c = n + 1$, $d = m$, we obtain

$$F_{n+m} = F_n F_{m-1} + F_{n+1} F_m$$

- With $a = n + 1$, $b = n - 1$, $c = n$, $d = n$, we obtain the well-known Cassini's identity

$$F_{n-1} F_{n+1} - F_n^2 = (-1)^n.$$

Outline

Motivation

A problem in the first issue of Fibonacci Quarterly in 1963 states:

H-8. Proposed by Brother U. Alfred, St. Mary's College, Calif.

Prove

$$\begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ F_{n+2}^2 & F_{n+3}^2 & F_{n+4}^2 \end{vmatrix} = 2(-1)^{n+1},$$

where F_n is the nth Fibonacci number.

Outline

Method 1: Laplace Expansion

Expand the cofactors along the first row of the matrix to obtain

$$D_n = F_n^2(F_{n+2}^2 F_{n+4}^2 - F_{n+3}^4) - F_{n+1}^2(F_{n+1}^2 F_{n+4}^2 - F_{n+2}^2 F_{n+3}^2) + F_{n+2}^2(F_{n+1}^2 F_{n+3}^2 - F_{n+2}^4)$$

Method 1: Laplace Expansion

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$$\begin{aligned}
D_n &= F_n^2(F_{n+2}^2 F_{n+4}^2 - F_{n+3}^4) - F_{n+1}^2(F_{n+1}^2 F_{n+4}^2 - F_{n+2}^2 F_{n+3}^2) \\
&\quad + F_{n+2}^2(F_{n+1}^2 F_{n+3}^2 - F_{n+2}^4) \\
&= F_n^2(F_{n+2}F_{n+4} - F_{n+3}^2)(F_{n+2}F_{n+4} + F_{n+3}^2) \\
&\quad - F_{n+1}^2(F_{n+1}F_{n+4} - F_{n+2}F_{n+3})(F_{n+1}F_{n+4} + F_{n+2}F_{n+3}) \\
&\qquad + F_{n+2}^2(F_{n+1}F_{n+3} - F_{n+2}^2)(F_{n+1}F_{n+3} + F_{n+2}^2)
\end{aligned}$$

Method 1: Laplace Expansion

Expand the cofactors along the first row of the matrix to obtain

$$\begin{aligned}
D_n &= F_n^2(F_{n+2}^2F_{n+4}^2 - F_{n+3}^4) - F_{n+1}^2(F_{n+1}^2F_{n+4}^2 - F_{n+2}^2F_{n+3}^2) \\
&\quad + F_{n+2}^2(F_{n+1}^2F_{n+3}^2 - F_{n+2}^4) \\
&= F_n^2(F_{n+2}F_{n+4} - F_{n+3}^2)(F_{n+2}F_{n+4} + F_{n+3}^2) \\
&\quad - F_{n+1}^2(F_{n+1}F_{n+4} - F_{n+2}F_{n+3})(F_{n+1}F_{n+4} + F_{n+2}F_{n+3}) \\
&\quad + F_{n+2}^2(F_{n+1}F_{n+3} - F_{n+2}^2)(F_{n+1}F_{n+3} + F_{n+2}^2) \\
&= F_n^2 \cdot (-1)^{n+1} \cdot (F_{n+2}F_{n+4} + F_{n+3}^2) \\
&\quad - F_{n+1}^2 \cdot (-1)^n \cdot F_1F_2 \cdot (F_{n+1}F_{n+4} + F_{n+2}F_{n+3}) \\
&\quad F_{n+2}^2 \cdot (-1)^n \cdot (F_{n+1}F_{n+3} + F_{n+2}^2)
\end{aligned}$$

Method 1: Laplace Expansion

Expand the cofactors along the first row of the matrix to obtain

$$
\begin{aligned}
D_n &= F_n^2(F_{n+2}^2F_{n+4}^2 - F_{n+3}^4) - F_{n+1}^2(F_{n+1}^2F_{n+4}^2 - F_{n+2}^2F_{n+3}^2) \\
&\quad + F_{n+2}^2(F_{n+1}^2F_{n+3}^2 - F_{n+2}^4) \\
&= F_n^2(F_{n+2}F_{n+4} - F_{n+3}^2)(F_{n+2}F_{n+4} + F_{n+3}^2) \\
&\quad - F_{n+1}^2(F_{n+1}F_{n+4} - F_{n+2}F_{n+3})(F_{n+1}F_{n+4} + F_{n+2}F_{n+3}) \\
&\qquad + F_{n+2}^2(F_{n+1}F_{n+3} - F_{n+2}^2)(F_{n+1}F_{n+3} + F_{n+2}^2) \\
&= F_n^2 \cdot (-1)^{n+1} \cdot (F_{n+2}F_{n+4} + F_{n+3}^2) \\
&\quad - F_{n+1}^2 \cdot (-1)^n \cdot F_1F_2 \cdot (F_{n+1}F_{n+4} + F_{n+2}F_{n+3}) \\
&\qquad F_{n+2}^2 \cdot (-1)^n \cdot (F_{n+1}F_{n+3} + F_{n+2}^2) \\
&= (-1)^{n+1}[F_n^2(F_{n+2}F_{n+4} + F_{n+3}^2) + F_{n+1}^2(F_{n+1}F_{n+4} + F_{n+2}F_{n+3})) \\
&\quad - F_{n+2}^2(F_{n+1}F_{n+3} + F_{n+2}^2)]
\end{aligned}
$$

Method 1: Laplace Expansion (cont.)

$$D_n = (-1)^{n+1}[F_n^2(2F_{n+3}^2 + (-1)^{n+1}) + F_{n+1}^2(2F_{n+2}F_{n+3} + (-1)^n)) - F_{n+2}^2(2F_{n+2}^2 + (-1)^n)]$$

Method 1: Laplace Expansion (cont.)

$$\begin{aligned} D_n &= (-1)^{n+1}[F_n^2(2F_{n+3}^2 + (-1)^{n+1}) + F_{n+1}^2(2F_{n+2}F_{n+3} + (-1)^n)) \\ &\quad - F_{n+2}^2(2F_{n+2}^2 + (-1)^n)] \\ &= (-1)^{n+1}[(-1)^n(-F_n^2 + F_{n+1}^2 - F_{n+2}^2) \\ &\qquad + 2(F_n^2F_{n+3}^2 + F_{n+1}^2F_{n+2}F_{n+3} - F_{n+2}^4)] \end{aligned}$$

Method 1: Laplace Expansion (cont.)

$$\begin{aligned} D_n &= (-1)^{n+1}[F_n^2(2F_{n+3}^2 + (-1)^{n+1}) + F_{n+1}^2(2F_{n+2}F_{n+3} + (-1)^n)) \\ &\quad - F_{n+2}^2(2F_{n+2}^2 + (-1)^n)] \\ &= (-1)^{n+1}[(-1)^n(-F_n^2 + F_{n+1}^2 - F_{n+2}^2) \\ &\qquad\qquad + 2(F_n^2F_{n+3}^2 + F_{n+1}^2F_{n+2}F_{n+3} - F_{n+2}^4)] \\ &= (-1)^{n+1} \cdot 2[(-1)^{n+1}F_nF_{n+2} + (F_n^2F_{n+3}^2 - F_{n+2}^4) + F_{n+1}^2F_{n+2}F_{n+3}] \end{aligned}$$

Method 1: Laplace Expansion (cont.)

$$\begin{aligned}
D_n &= (-1)^{n+1}[F_n^2(2F_{n+3}^2 + (-1)^{n+1}) + F_{n+1}^2(2F_{n+2}F_{n+3} + (-1)^n)) \\
&\quad - F_{n+2}^2(2F_{n+2}^2 + (-1)^n)] \\
&= (-1)^{n+1}[(-1)^n(-F_n^2 + F_{n+1}^2 - F_{n+2}^2) \\
&\qquad\qquad\qquad + 2(F_n^2F_{n+3}^2 + F_{n+1}^2F_{n+2}F_{n+3} - F_{n+2}^4)] \\
&= (-1)^{n+1} \cdot 2[(-1)^{n+1}F_nF_{n+2} + (F_n^2F_{n+3}^2 - F_{n+2}^4) + F_{n+1}^2F_{n+2}F_{n+3}] \\
&= (-1)^{n+1} \cdot 2[(-1)^{n+1}F_nF_{n+2} + (F_nF_{n+3} - F_{n+2}^2)(F_nF_{n+3} + F_{n+2}^2) \\
&\quad + F_{n+1}^2F_{n+2}F_{n+3}]
\end{aligned}$$

Method 1: Laplace Expansion (cont.)

$$
\begin{aligned}
D_n &= (-1)^{n+1}[F_n^2(2F_{n+3}^2 + (-1)^{n+1}) + F_{n+1}^2(2F_{n+2}F_{n+3} + (-1)^n)) \\
&\quad - F_{n+2}^2(2F_{n+2}^2 + (-1)^n)] \\
&= (-1)^{n+1}[(-1)^n(-F_n^2 + F_{n+1}^2 - F_{n+2}^2) \\
&\qquad + 2(F_n^2F_{n+3}^2 + F_{n+1}^2F_{n+2}F_{n+3} - F_{n+2}^4)] \\
&= (-1)^{n+1} \cdot 2[(-1)^{n+1}F_nF_{n+2} + (F_n^2F_{n+3}^2 - F_{n+2}^4) + F_{n+1}^2F_{n+2}F_{n+3}] \\
&= (-1)^{n+1} \cdot 2[(-1)^{n+1}F_nF_{n+2} + (F_nF_{n+3} - F_{n+2}^2)(F_nF_{n+3} + F_{n+2}^2) \\
&\quad + F_{n+1}^2F_{n+2}F_{n+3}] \\
&= (-1)^{n+1} \cdot 2[(-1)^{n+1}F_nF_{n+2} + F_{n+1}^2F_{n+2}F_{n+3} \\
&\quad + (F_nF_{n+3} - F_{n+1}F_{n+3} + (-1)^n)(F_nF_{n+3} + F_{n+1}F_{n+3} + (-1)^{n+1})]
\end{aligned}
$$

Method 1: Laplace Expansion (cont.)

$$D_n = (-1)^{n+1} \cdot 2[(-1)^{n+1} F_n F_{n+2} + (-1)^n F_{n-1} F_{n+3} + (-1)^n F_{n+2} F_{n+3} \\ + F_{n+1}^2 F_{n+2} F_{n+3} - F_{n-1} F_{n+2} F_{n+3}^2 - 1]$$

Method 1: Laplace Expansion (cont.)

$$
\begin{aligned}
D_n &= (-1)^{n+1} \cdot 2[(-1)^{n+1} F_n F_{n+2} + (-1)^n F_{n-1} F_{n+3} + (-1)^n F_{n+2} F_{n+3} \\
&\quad + F_{n+1}^2 F_{n+2} F_{n+3} - F_{n-1} F_{n+2} F_{n+3}^2 - 1] \\
&= (-1)^{n+1} \cdot 2[(-1)^n (F_{n+3} F_{n-1} - F_n F_{n+2}) + F_{n+2} F_{n+3} (F_{n+1}^2 + (-1)^n) \\
&\quad - F_{n-1} F_{n+2} F_{n+3}^2 - 1]
\end{aligned}
$$

Method 1: Laplace Expansion (cont.)

$$
\begin{aligned}
D_n &= (-1)^{n+1} \cdot 2[(-1)^{n+1} F_n F_{n+2} + (-1)^n F_{n-1} F_{n+3} + (-1)^n F_{n+2} F_{n+3} \\
&\quad + F_{n+1}^2 F_{n+2} F_{n+3} - F_{n-1} F_{n+2} F_{n+3}^2 - 1] \\
&= (-1)^{n+1} \cdot 2[(-1)^n (F_{n+3} F_{n-1} - F_n F_{n+2}) + F_{n+2} F_{n+3} (F_{n+1}^2 + (-1)^n) \\
&\quad - F_{n-1} F_{n+2} F_{n+3}^2 - 1] \\
&= (-1)^{n+1} \cdot 2[(-1)^n \cdot (-1)^n \cdot F_3 F_1 - 1 \\
&\qquad\qquad + F_{n+2} F_{n+3} ((-1)^n + F_{n+1}^2 - F_{n-1} F_{n+3})]
\end{aligned}
$$

Method 1: Laplace Expansion (cont.)

$$
\begin{aligned}
D_n &= (-1)^{n+1} \cdot 2[(-1)^{n+1} F_n F_{n+2} + (-1)^n F_{n-1} F_{n+3} + (-1)^n F_{n+2} F_{n+3} \\
&\quad + F_{n+1}^2 F_{n+2} F_{n+3} - F_{n-1} F_{n+2} F_{n+3}^2 - 1] \\
&= (-1)^{n+1} \cdot 2[(-1)^n (F_{n+3} F_{n-1} - F_n F_{n+2}) + F_{n+2} F_{n+3} (F_{n+1}^2 + (-1)^n) \\
&\quad - F_{n-1} F_{n+2} F_{n+3}^2 - 1] \\
&= (-1)^{n+1} \cdot 2[(-1)^n \cdot (-1)^n \cdot F_3 F_1 - 1 \\
&\qquad\qquad + F_{n+2} F_{n+3} ((-1)^n + F_{n+1}^2 - F_{n-1} F_{n+3})] \\
&= \boxed{(-1)^{n+1} \cdot 2}
\end{aligned}
$$

Outline

Method 2: Recurrence Relation

Theorem

The squares of the Fibonacci numbers satisfy the linear homogeneous recurrence relation

$$F_{n+3}^2 = 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2.$$

Method 2: Recurrence Relation (cont.)

Substitute this recurrence relation for the last row of the determinant to obtain

$$D_n = \begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ 2F_{n+1}^2 + 2F_n^2 - F_{n-1}^2 & 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2 & 2F_{n+3}^2 + 2F_{n+2}^2 - F_{n+1}^2 \end{vmatrix}.$$

Method 2: Recurrence Relation (cont.)

Apply standard row operations to obtain

$$D_n = \begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ -F_{n-1}^2 & -F_n^2 & -F_{n+1}^2 \end{vmatrix} = - \begin{vmatrix} F_{n-1}^2 & F_n^2 & F_{n+1}^2 \\ F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \end{vmatrix} = -D_{n-1}.$$

It follows that $D_n = (-1)^{n+1} D_1$. Since $D_1 = 2$, we have

$$\boxed{D_n = (-1)^{n+1} 2.}$$

General Relations

$$F_{n+2} = F_{n+1} + F_n$$

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$$F_{n+3}^2 = 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2$$

General Relations

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_n \\ F_{n+3}^2 &= 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2 \\ F_{n+4}^3 &= 3F_{n+3}^3 + 6F_{n+2}^3 - 3F_{n+1}^3 - F_n^3 \end{aligned}$$

General Relations

$$F_{n+2} = F_{n+1} + F_n$$

$$F_{n+3}^2 = 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2$$

$$F_{n+4}^3 = 3F_{n+3}^3 + 6F_{n+2}^3 - 3F_{n+1}^3 - F_n^3$$

$$F_{n+5}^4 = 5F_{n+4}^4 + 15F_{n+3}^4 - 15F_{n+2}^4 - 5F_{n+1}^4 + F_n^4$$

General Relations

$$\begin{aligned}
F_{n+2} &= F_{n+1} + F_n \\
F_{n+3}^2 &= 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2 \\
F_{n+4}^3 &= 3F_{n+3}^3 + 6F_{n+2}^3 - 3F_{n+1}^3 - F_n^3 \\
F_{n+5}^4 &= 5F_{n+4}^4 + 15F_{n+3}^4 - 15F_{n+2}^4 - 5F_{n+1}^4 + F_n^4 \\
F_{n+6}^5 &= 8F_{n+5}^5 + 40F_{n+4}^5 - 60F_{n+3}^5 - 40F_{n+2}^5 + 8F_{n+1}^5 + F_n^5
\end{aligned}$$

General Relations

$$
\begin{aligned}
F_{n+2} &= F_{n+1} + F_n \\
F_{n+3}^2 &= 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2 \\
F_{n+4}^3 &= 3F_{n+3}^3 + 6F_{n+2}^3 - 3F_{n+1}^3 - F_n^3 \\
F_{n+5}^4 &= 5F_{n+4}^4 + 15F_{n+3}^4 - 15F_{n+2}^4 - 5F_{n+1}^4 + F_n^4 \\
F_{n+6}^5 &= 8F_{n+5}^5 + 40F_{n+4}^5 - 60F_{n+3}^5 - 40F_{n+2}^5 + 8F_{n+1}^5 + F_n^5 \\
F_{n+7}^6 &= 13F_{n+6}^6 + 104F_{n+5}^6 - 260F_{n+4}^6 - 260F_{n+3}^6 \\
&\qquad +104F_{n+2}^6 + 13F_{n+1}^6 - F_n^6
\end{aligned}
$$

Fibonomial Coefficient and Fibonomial Triangle

$\binom{p}{k}_F$ is called the Fibonomial coefficient and is defined by

$$\binom{p}{k}_F = \frac{F_p F_{p-1} \cdots F_{p-k+1}}{F_k F_{k-1} \cdots F_1}.$$

$n=0$								1							
$n=1$							1		1						
$n=2$						1		1		1					
$n=3$					1		2		2		1				
$n=4$				1		3		6		3		1			
$n=5$			1		5		15		15		5		1		
$n=6$		1		8		40		60		40		8		1	
$n=7$	1		13		104		260		260		104		13		1

Properties of Fibonomials

1.

$$\boxed{\binom{n}{k}_F = \binom{n}{n-k}_F}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

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1. $$\boxed{\binom{n}{k}_F = \binom{n}{n-k}_F}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

2. $$\boxed{\binom{n}{k}_F = F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Properties of Fibonomials (cont.)

3.

$$\boxed{\binom{n}{k}_F = \frac{F_{n-k+1}}{F_k}\binom{n}{k-1}_F, \quad k \neq 0}$$

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}, \quad k \neq 0$$

Properties of Fibonomials (cont.)

3. $$\boxed{\binom{n}{k}_F = \frac{F_{n-k+1}}{F_k}\binom{n}{k-1}_F, \quad k \neq 0}$$

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}, \quad k \neq 0$$

4. $$\boxed{\binom{n}{k}_F = \frac{F_n}{F_{n-k}}\binom{n-1}{k}_F, \quad k \neq n}$$

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}, \quad k \neq n$$

General Relations (cont.)

Theorem (Riordan, 1962)

The powers of the Fibonacci numbers satisfy the linear homogeneous recurrence relation

$$S_n^{(p-1)} := \sum_{k=0}^{p} \binom{p}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} F_{n+p-k}^{p-1} = 0,$$

for all $n = 1, 2, \ldots$.

▸ **29.** [*M23*] (*Fibonomial coefficients.*) Édouard Lucas defined the quantities

$$\binom{n}{k}_{\mathcal{F}} = \frac{F_n F_{n-1} \ldots F_{n-k+1}}{F_k F_{k-1} \ldots F_1} = \prod_{j=1}^{k} \left(\frac{F_{n-k+j}}{F_j} \right)$$

in a manner analogous to binomial coefficients. (a) Make a table of $\binom{n}{k}_{\mathcal{F}}$ for $0 \le k \le n \le 6$. (b) Show that $\binom{n}{k}_{\mathcal{F}}$ is always an integer because we have

$$\binom{n}{k}_{\mathcal{F}} = F_{k-1} \binom{n-1}{k}_{\mathcal{F}} + F_{n-k+1} \binom{n-1}{k-1}_{\mathcal{F}}.$$

▸ **30.** [*M38*] (D. Jarden, T. Motzkin.) The sequence of mth powers of Fibonacci numbers satisfies a recurrence relation in which each term depends on the preceding $m+1$ terms. Show that

$$\sum_k \binom{m}{k}_{\mathcal{F}} (-1)^{\lceil (m-k)/2 \rceil} F_{n+k}^{m-1} = 0, \quad \text{if } m > 0.$$

For example, when $m = 3$ we get the identity $F_n^2 - 2F_{n+1}^2 - 2F_{n+2}^2 + F_{n+3}^2 = 0$.

Case $p = 3$

From

$$S_n^{(1)} = F_{n+2} - F_{n+1} - F_n = 0 \quad \text{and} \quad S_{n+1}^{(1)} = F_{n+3} - F_{n+2} - F_{n+1} = 0,$$

we have

$$S_n^{(2)} = F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2$$

Case $p = 3$

From

$$S_n^{(1)} = F_{n+2} - F_{n+1} - F_n = 0 \quad \text{and} \quad S_{n+1}^{(1)} = F_{n+3} - F_{n+2} - F_{n+1} = 0,$$

we have

$$\begin{aligned} S_n^{(2)} &= F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 \\ &= F_{n+3} \cdot F_{n+3} - 2F_{n+2} \cdot F_{n+2} - 2F_{n+1}^2 + F_n^2 \end{aligned}$$

Case $p = 3$

From

$$S_n^{(1)} = F_{n+2} - F_{n+1} - F_n = 0 \quad \text{and} \quad S_{n+1}^{(1)} = F_{n+3} - F_{n+2} - F_{n+1} = 0,$$

we have

$$\begin{aligned} S_n^{(2)} &= F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 \\ &= F_{n+3} \cdot F_{n+3} - 2F_{n+2} \cdot F_{n+2} - 2F_{n+1}^2 + F_n^2 \\ &= (2F_{n+1} + F_n)F_{n+3} - 2(F_{n+1} + F_n)F_{n+2} - 2F_{n+1}^2 + F_n^2 \end{aligned}$$

Case $p = 3$

From

$$S_n^{(1)} = F_{n+2} - F_{n+1} - F_n = 0 \quad \text{and} \quad S_{n+1}^{(1)} = F_{n+3} - F_{n+2} - F_{n+1} = 0,$$

we have

$$\begin{aligned} S_n^{(2)} &= F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 \\ &= F_{n+3} \cdot F_{n+3} - 2F_{n+2} \cdot F_{n+2} - 2F_{n+1}^2 + F_n^2 \\ &= (2F_{n+1} + F_n)F_{n+3} - 2(F_{n+1} + F_n)F_{n+2} - 2F_{n+1}^2 + F_n^2 \\ &= F_{n+1}(2F_{n+3} - 2F_{n+2} - 2F_{n+1}) + F_n(F_{n+3} - 2F_{n+2} + F_n) \end{aligned}$$

Case $p = 3$

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$$\begin{aligned} S_n^{(2)} &= F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 \\ &= F_{n+3} \cdot F_{n+3} - 2F_{n+2} \cdot F_{n+2} - 2F_{n+1}^2 + F_n^2 \\ &= (2F_{n+1} + F_n)F_{n+3} - 2(F_{n+1} + F_n)F_{n+2} - 2F_{n+1}^2 + F_n^2 \\ &= F_{n+1}(2F_{n+3} - 2F_{n+2} - 2F_{n+1}) + F_n(F_{n+3} - 2F_{n+2} + F_n) \\ &= F_{n+1} \cdot 2S_{n+1}^{(1)} + F_n \cdot (S_{n+1}^{(1)} - S_n^{(1)}) \end{aligned}$$

Case $p = 3$

From

$$S_n^{(1)} = F_{n+2} - F_{n+1} - F_n = 0 \quad \text{and} \quad S_{n+1}^{(1)} = F_{n+3} - F_{n+2} - F_{n+1} = 0,$$

we have

$$\begin{aligned}
S_n^{(2)} &= F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 \\
&= F_{n+3} \cdot F_{n+3} - 2F_{n+2} \cdot F_{n+2} - 2F_{n+1}^2 + F_n^2 \\
&= (2F_{n+1} + F_n)F_{n+3} - 2(F_{n+1} + F_n)F_{n+2} - 2F_{n+1}^2 + F_n^2 \\
&= F_{n+1}(2F_{n+3} - 2F_{n+2} - 2F_{n+1}) + F_n(F_{n+3} - 2F_{n+2} + F_n) \\
&= F_{n+1} \cdot 2S_{n+1}^{(1)} + F_n \cdot (S_{n+1}^{(1)} - S_n^{(1)}) \\
&= F_{n+1}(0) + F_n(0) = \boxed{0}
\end{aligned}$$

Case $p = 4$

From

$$S_n^{(2)} = F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0$$

and

$$S_{n+1}^{(2)} = F_{n+4}^2 - 2F_{n+3}^2 - 2F_{n+2}^2 + F_{n+1}^2 = 0,$$

we have

$$S_n^{(3)} = F_{n+4}^3 - 3F_{n+3}^3 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3$$

Case $p = 4$

From

$$S_n^{(2)} = F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0$$

and

$$S_{n+1}^{(2)} = F_{n+4}^2 - 2F_{n+3}^2 - 2F_{n+2}^2 + F_{n+1}^2 = 0,$$

we have

$$\begin{aligned} S_n^{(3)} &= F_{n+4}^3 - 3F_{n+3}^3 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3 \\ &= F_{n+4} \cdot F_{n+4}^2 - 3F_{n+3} \cdot F_{n+3}^2 - 6F_{n+2} \cdot F_{n+2}^2 + 3F_{n+1}^3 + F_n^3 \end{aligned}$$

Case $p = 4$

From

$$S_n^{(2)} = F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0$$

and

$$S_{n+1}^{(2)} = F_{n+4}^2 - 2F_{n+3}^2 - 2F_{n+2}^2 + F_{n+1}^2 = 0,$$

we have

$$\begin{aligned} S_n^{(3)} &= F_{n+4}^3 - 3F_{n+3}^3 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3 \\ &= F_{n+4} \cdot F_{n+4}^2 - 3F_{n+3} \cdot F_{n+3}^2 - 6F_{n+2} \cdot F_{n+2}^2 + 3F_{n+1}^3 + F_n^3 \\ &= (3F_{n+1} + 2F_n)F_{n+4}^2 - 3(2F_{n+1} + F_n)F_{n+3}^2 \\ &\qquad\qquad - 6(F_{n+1} + F_n)F_{n+2}^2 + 3F_{n+1}^3 + F_n^3 \end{aligned}$$

Case $p = 4$

From

$$S_n^{(2)} = F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0$$

and

$$S_{n+1}^{(2)} = F_{n+4}^2 - 2F_{n+3}^2 - 2F_{n+2}^2 + F_{n+1}^2 = 0,$$

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$$\begin{aligned} S_n^{(3)} &= F_{n+4}^3 - 3F_{n+3}^3 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3 \\ &= F_{n+4} \cdot F_{n+4}^2 - 3F_{n+3} \cdot F_{n+3}^2 - 6F_{n+2} \cdot F_{n+2}^2 + 3F_{n+1}^3 + F_n^3 \\ &= (3F_{n+1} + 2F_n)F_{n+4}^2 - 3(2F_{n+1} + F_n)F_{n+3}^2 \\ &\qquad - 6(F_{n+1} + F_n)F_{n+2}^2 + 3F_{n+1}^3 + F_n^3 \\ &= F_{n+1}(3F_{n+4}^2 - 6F_{n+3}^2 - 6F_{n+2}^2 - 3F_{n+1}^2) \\ &\qquad + F_n(2F_{n+4}^2 - 3F_{n+3}^2 - 6F_{n+2}^2 + F_n^2) \end{aligned}$$

Case $p = 4$

From

$$S_n^{(2)} = F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0$$

and

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we have

$$\begin{aligned}
S_n^{(3)} &= F_{n+4}^3 - 3F_{n+3}^3 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3 \\
&= F_{n+4} \cdot F_{n+4}^2 - 3F_{n+3} \cdot F_{n+3}^2 - 6F_{n+2} \cdot F_{n+2}^2 + 3F_{n+1}^3 + F_n^3 \\
&= (3F_{n+1} + 2F_n)F_{n+4}^2 - 3(2F_{n+1} + F_n)F_{n+3}^2 \\
&\qquad - 6(F_{n+1} + F_n)F_{n+2}^2 + 3F_{n+1}^3 + F_n^3 \\
&= F_{n+1}(3F_{n+4}^2 - 6F_{n+3}^2 - 6F_{n+2}^2 - 3F_{n+1}^2) \\
&\qquad + F_n(2F_{n+4}^2 - 3F_{n+3}^2 - 6F_{n+2}^2 + F_n^2) \\
&= F_{n+1} \cdot 3S_{n+1}^{(2)} + F_n \cdot (2S_{n+1}^{(2)} + S_n^{(2)})
\end{aligned}$$

Case $p = 4$

From

$$S_n^{(2)} = F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0$$

and

$$S_{n+1}^{(2)} = F_{n+4}^2 - 2F_{n+3}^2 - 2F_{n+2}^2 + F_{n+1}^2 = 0,$$

we have

$$\begin{aligned}
S_n^{(3)} &= F_{n+4}^3 - 3F_{n+3}^3 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3 \\
&= F_{n+4} \cdot F_{n+4}^2 - 3F_{n+3} \cdot F_{n+3}^2 - 6F_{n+2} \cdot F_{n+2}^2 + 3F_{n+1}^3 + F_n^3 \\
&= (3F_{n+1} + 2F_n)F_{n+4}^2 - 3(2F_{n+1} + F_n)F_{n+3}^2 \\
&\qquad - 6(F_{n+1} + F_n)F_{n+2}^2 + 3F_{n+1}^3 + F_n^3 \\
&= F_{n+1}(3F_{n+4}^2 - 6F_{n+3}^2 - 6F_{n+2}^2 - 3F_{n+1}^2) \\
&\qquad + F_n(2F_{n+4}^2 - 3F_{n+3}^2 - 6F_{n+2}^2 + F_n^2) \\
&= F_{n+1} \cdot 3S_{n+1}^{(2)} + F_n \cdot (2S_{n+1}^{(2)} + S_n^{(2)}) \\
&= F_{n+1}(0) + F_n(0) = \boxed{0}
\end{aligned}$$

Case $p \geq 5$

From

$$S_n^{(p-1)} = \sum_{k=0}^{p} \binom{p}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} F_{n+p-k}^{p-1} = 0,$$

we have

$$\begin{aligned} S_n^{(p)} &= \sum_{k=0}^{p+1} \binom{p+1}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} F_{n+p+1-k}^{p} \\ &= \sum_{k=0}^{p+1} \binom{p+1}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} F_{n+p+1-k} \cdot F_{n+p+1-k}^{p-1} \\ &= \sum_{k=0}^{p+1} \binom{p+1}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} (F_{p+1-k} F_{n+1} + F_{p-k} F_n) \cdot F_{n+p+1-k}^{p-1} \end{aligned}$$

Case $p \geq 5$ (cont.)

$$
\begin{aligned}
S_n^{(p)} &= F_{n+1} \cdot \sum_{k=0}^{p+1} \binom{p+1}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} F_{p+1-k} \cdot F_{n+p+1-k}^{p-1} \\
&\qquad + F_n \cdot \sum_{k=0}^{p+1} \binom{p+1}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} F_{p-k} \cdot F_{n+p+1-k}^{p-1} \\
&\overset{(*)}{=} F_{n+1} \cdot F_{p+1} S_{n+1}^{(p-1)} + F_n \cdot (F_p \, S_{n+1}^{(p-1)} + (-1)^{p-1} S_n^{(p-1)}) \\
&= F_{n+1}(0) + F_n(0) = \boxed{0}
\end{aligned}
$$

Some Lemmas

Lemma

Let k be an integer. Then

$$\left\lceil \frac{k}{2} \right\rceil + \left\lceil \frac{k-1}{2} \right\rceil = k.$$

Proof. Work on congruence class modulo 4.

Lemma

$$(-1)^{n+k}\binom{n}{k-1}_F = F_n\binom{n}{k}_F - F_{n-k}\binom{n+1}{k}_F$$

Proof. Expand the Fibonomial coefficients and apply the identity

$$F_{n+1}F_{n-k} - F_nF_{n-k+1} = (-1)^{n-k+1}F_k.$$

Verification of Equality (*)

$$
\begin{aligned}
&F_p\, S_{n+1}^{(p-1)} + (-1)^{p-1} S_n^{(p-1)} \\
&= F_p \cdot \sum_{k=0}^{p} \binom{p}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} F_{n+p+1-k}^{p-1} \\
&\qquad\qquad + (-1)^{p-1} \cdot \sum_{k=0}^{p} \binom{p}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} F_{n+p-k}^{p-1} \\
&= F_p \cdot \sum_{k=0}^{p} \binom{p}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} F_{n+p+1-k}^{p-1} \\
&\qquad\qquad + (-1)^{p-1} \cdot \sum_{k=1}^{p+1} \binom{p}{k-1}_F (-1)^{\lceil \frac{k-1}{2} \rceil} F_{n+p+1-k}^{p-1} \\
&= \sum_{k=0}^{p+1} \left(\binom{p}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} F_p + \binom{p}{k-1}_F (-1)^{p-1+\lceil \frac{k-1}{2} \rceil} \right) F_{n+p+1-k}^{p-1}
\end{aligned}
$$

Verification of Equality (*) (cont.)

$$
\begin{aligned}
&F_p\, S_{n+1}^{(p-1)} + (-1)^{p-1} S_n^{(p-1)} \\
&= \sum_{k=0}^{p+1} \left(\binom{p}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} F_p + \binom{p}{k-1}_F (-1)^{p-1+k-\lceil \frac{k}{2} \rceil} \right) F_{n+p+1-k}^{p-1} \\
&= \sum_{k=0}^{p+1} \left(\binom{p}{k}_F F_p - \binom{p}{k-1}_F (-1)^{p+k} \right) (-1)^{\lceil \frac{k}{2} \rceil} F_{n+p+1-k}^{p-1} \\
&= \sum_{k=0}^{p+1} \binom{p+1}{k}_F (-1)^{\lceil \frac{k}{2} \rceil} F_{p-k} \cdot F_{n+p+1-k}^{p-1}
\end{aligned}
$$

Verification of Equality (*) (cont.)

$$\begin{aligned}
F_{p+1}S_{n+1}^{(p-1)} &= F_{p+1}\sum_{k=0}^{p}\binom{p}{k}_F(-1)^{\lceil\frac{k}{2}\rceil}F_{n+p+1-k}^{p-1} \\
&= F_{p+1}\sum_{k=0}^{p+1}\binom{p}{k}_F(-1)^{\lceil\frac{k}{2}\rceil}F_{n+p+1-k}^{p-1} \\
&= \sum_{k=0}^{p+1}\binom{p}{k}_F(-1)^{\lceil\frac{k}{2}\rceil}F_{p+1}\cdot F_{n+p+1-k}^{p-1} \\
&= \sum_{k=0}^{p+1}\binom{p+1}{k}_F(-1)^{\lceil\frac{k}{2}\rceil}F_{p+1-k}\cdot F_{n+p+1-k}^{p-1}
\end{aligned}$$

Outline

Desnanot-Jacobi Identity

Theorem

Let A be $n \times n$ matrix. Then

$$\det(A)\det(A_{1,n}^{1,n}) = \det(A_1^1)\det(A_n^n) - \det(A_1^n)\det(A_n^1).$$

Desnanot-Jacobi Identity

Theorem

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$$\det(A)\det(A_{1,n}^{1,n}) = \det(A_1^1)\det(A_n^n) - \det(A_1^n)\det(A_n^1).$$

For example, let $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 4 & 4 \\ 2 & -1 & 1 \end{bmatrix}$. Then

$$A_{1,3}^{1,3} = [4] \implies \det(A_{1,3}^{1,3}) = 4, \quad A_1^1 = \begin{bmatrix} 4 & 4 \\ -1 & 1 \end{bmatrix} \implies \det(A_1^1) = 8$$

$$A_3^3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \implies \det(A_3^3) = -2, \quad A_1^3 = \begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix} \implies \det(A_1^3) = -11$$

$$A_3^1 = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix} \implies \det(A_3^1) = 20$$

Desnanot-Jacobi Identity

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Let A be $n \times n$ matrix. Then

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$$A_3^1 = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix} \implies \det(A_3^1) = 20$$

Hence, $\det(A) = \frac{(8)(-2)-(-11)(20)}{4} = 51$.

Theorem (T., Thanatipanonda, 2016)

Let $D(r,s,k,n) = \left|F^r_{s+k(n+i+j)}\right|$, $0 \le i,j \le r$. *Then*

$$D(r,s,k,n) = (-1)^{(s+kn+1)\binom{r+1}{2}}(F_k^r F_{2k}^{r-1} \cdots F_{rk})^2 \cdot \prod_{i=0}^{r} \binom{r}{i}.$$