Generalizing the First Partition Identity and Magic Configurations

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Partitions

A partition of n is an unordered collection of positive integers whose sum is n.

$$P(4) = \{4, 31, 22, 211, 1111\}, \quad p(4) = 5$$

Serious study of partitions starts with Euler, resurgence in late 1800s, major names include Sylvester, Ramanujan, Freeman Dyson, George Andrews, Ken Ono.

Tools include generating functions, combinatorial proofs, and modular forms.

Odds & Distincts

One line of study asks why two restricted types of partitions have the same count. E.g., for each n, there are an equal number of partitions into odd parts and partitions into distinct parts.

n	distinct	odd	#
1	1	1	1
2	2	11	1
3	3, 21	3, 1^3	2
4	4, 31	31, 1^4	2
5	5, 41, 32	5, 311, 1^5	3
6	6, 51, 42, 321	51, 33, 31^3, 1^6	4
7	7, 61, 52, 43, 421	7, 511, 331, 31^4, 1^7	5
8	8, 71, 62, 53, 521, 431	71, 53, 51^3, 3311, 31^5, 1^8	6

Generating Functions

We can move the addition of parts into the combination of exponents in an infinite polynomial.

$$(1 + x^1 + x^{1+1} + \cdots)(1 + x^2 + x^{2+2} + \cdots)(1 + x^3 + x^{3+3} + \cdots)\cdots$$
$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + \cdots$$

$$5x^4 = x^4 + x^3x^1 + x^{2+2} + x^2x^{1+1} + x^{1+1+1+1}$$

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots = p(0)x^0 + p(1)x^1 + p(2)x^2 + p(3)x^3 + \cdots$$

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Euler's Proof

Odd parts: $(1 + x^1 + x^{1+1} + \cdots)(1 + x^3 + \cdots)(1 + x^5 + \cdots)\cdots$
Distinct parts: $(1 + x^1)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)\cdots$

Euler's Proof

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Distinct parts: $(1 + x^1)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)\cdots$

$$\begin{aligned}
\prod_k \frac{1}{1 - x^{2k-1}} &= \prod_k \frac{1}{1 - x^{2k-1}} \cdot \frac{1 - x^{2k}}{1 - x^{2k}} \\
&= \prod_\ell \frac{1 - x^{2\ell}}{1 - x^\ell} \\
&= \prod_\ell \frac{(1 - x^\ell)(1 + x^\ell)}{1 - x^\ell} \\
&= \prod_\ell (1 + x^\ell)
\end{aligned}$$

Glaisher's Proof

1883 combinatorial proof based on the operation

If there is some even part $2k$, replace it with k, k.

E.g., $42 \to 411 \to 2211 \to 21^4 \to 1^6$.

Applying this sufficiently many times removes any even parts, leaving a partition with only odd parts.

The reverse map, k, k to $2k$, eventually removes any repetition, leaving a partition with distinct parts. Thus, a bijection.

Glaisher's Operation

Glaisher's operation on all partitions of 6.

$$6 \rightarrow 33$$

$$51$$

$$321 \rightarrow 3111$$

$$42 \rightarrow 411 \rightarrow 2211 \rightarrow 21111 \rightarrow 111111$$

$$42 \searrow 222 \nearrow 2211$$

Generalizing the Identity

Several generalizations of the odd–distinct partition identity, e.g.,

$\#$(partitions of n with no part divisible by d)
$= \#$(partitions of n with no part repeated d or more times)

$\#$(partitions of n with parts all 1, 5, or 6 mod 8)
$= \#$(partitions of n with distinct parts, none 3 mod 4)

$\#$(partitions of n with odd parts all less than $2N$)
$= \#$(partitions of n all parts $\leq 2N$, parts $\leq N$ distinct)

Relaxing the Identity

Instead of adding constraints, relax the conditions. Consider partitions where one part repeats, where one part is even.

n	one repeated part	one even part	#
2	11	2	1
3	1^3	21	1
4	22, 211, 1^4	4, 22, 211	3
5	311, 221, 21^3, 1^5	41, 32, 221, 21^3	4
6	411, 33, 31^3, 2^3, 21^4, 1^6	6, 41, 321, 2^3, 2211, 21^4	6

George Andrews proved with generating functions.

Combinatorial Proof

Bijection: For repeated part r, merge r, r to $2r$, the even number in the corresponding partition. (There may be more r parts.) Split all other parts down to odd parts or $2r$.

$$411 \to 4\underline{2} \to 222$$
$$33 \to \underline{6}$$
$$3111 \to 3\underline{2}11$$
$$222 \to \underline{4}2 \to 411$$
$$21^4 \to 2\underline{2}11$$
$$1^6 \to \underline{2}1^4$$

George's former student Shishuo Fu (Chongxing) "Glaisherized" the results (i.e., odd–distinct is $d = 2$ case), bijection.

Another Equal Count

In the partitions into distinct parts and into odd parts, consider the total number of parts:

n	distinct		odd	

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n	distinct		odd	
1	1	1	1	1
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3	3, 21	3	3, 1^3	4
4	4, 31	3	31, 1^4	6
5	5, 41, 32	5	5, 311, 1^5	9
6	6, 51, 42, 321	8	51, 33, 31^3, 1^6	14
7	7, 61, 52, 43, 421	10	7, 511, 331, 31^4, 1^7	19
8	8, 71, 62, 53, 521, 431	13	71, 53, 51^3, 3311, 31^5, 1^8	26

Another equal count

n	# odd parts − # distinct parts	# "relaxed" partitions
1	$1-1$	0
2	$2-1$	1
3	$4-3$	1
4	$6-3$	3
5	$9-5$	4
6	$14-8$	6
7	$19-10$	9
8	$26-13$	13

Another equal count

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2	$2-1$	1
3	$4-3$	1
4	$6-3$	3
5	$9-5$	4
6	$14-8$	6
7	$19-10$	9
8	$26-13$	13

George Andrews proved with generating functions (tricky: differentiate to get number of parts).

Bijections typically between various partitions. Number of parts??

Glaisher on $P(6)$ organized by number of parts:

$$
\begin{array}{ccccccccccc}
6 & \rightarrow & 33 & & & & & & & & \\
 & & 51 & & & & & & & & \\
 & & & & 321 & \rightarrow & 3111 & & & & \\
 & & 42 & \rightarrow & 411 & \rightarrow & 2211 & \rightarrow & 21111 & \rightarrow & 111111 \\
 & & & \searrow & & \nearrow & & & & & \\
 & & & & 222 & & & & & &
\end{array}
$$

Insight

- Each application of Glaisher's split operation adds one part.
- # parts in odd partitions − # parts in distinct partitions = # applications of Glaisher between the two sets.
- Want to associate a partition with one part repeated to each application of Glaisher's map.

$6 \rightarrow \boxed{33}$

51

$321 \rightarrow \boxed{3111}$

$42 \rightarrow \boxed{411} \rightarrow 2211 \rightarrow \boxed{21111} \rightarrow \boxed{111111}$

$42 \searrow \boxed{222} \nearrow 2211$

Proof Ingredients

- Only powers of 2 matter: 16, $3 \cdot 16$, and $2017 \cdot 16$ all split the same way.
- From 2^j to 1^{2^j} takes $2^j - 1$ applications of the split map. Induction step: 2^j to $2^{j-1}, 2^{j-1}$, so $1 + 2(2^{j-1} - 1)$ splits.
- The poset from 2^j to 1^{2^j} has exactly $2^j - 1$ (binary) partitions with a single part repeated. Induction sketch: From 2^{j-1} to 2^j, use two copies of the partitions and one more.

  E.g., 4 has 22 211 1^4.

  8 then has 422 4211 41^4 / 2^4 21^6 1^8

  and 44 for $3 + 3 + 1 = 7$ partitions.

- For (distinct) multipart binary partitions, $\sum \delta_i 2^i$ has $\sum \delta_i (2^i - 1)$ partitions with one part repeated.
  Idea: Hold all but one part fixed, apply above procedure to the free part. Split any additional repetitions, maintaining the repeated part of the smaller partition.

E.g., 842 has $7 + 3 + 1 = 11$ partitions with one part repeated:

2 free	combine 84 and 11: 8411
4 free	combine 82 with 22, 211, 1^4: 8222, 82211 $\rightarrow 81^6, 821^4$
8 free	combine 42 with 44, 422, 4211, 41^4, 2^4, 21^6, 1^8:

$$4442, 44222 \rightarrow 2^7, 442211 \rightarrow 1^{14}, 44214^4 \rightarrow 21^{12}, 42^5, 4221^6 \rightarrow 41^{10}, 421^8$$

First Summary

- Write a general partition as $(d_1(b_1), d_2(b_2), \ldots)$ where the d_i are odd numbers and the b_i are binary partitions. Reason as before for each b_i, carrying the odd factor.

$$\begin{aligned}&\#\text{ parts in odd partitions} - \#\text{ parts in distinct partitions}\\ &\quad = \#\text{ applications of Glaisher between the two sets}\\ &\quad = \#\text{ partitions with a single repeated value}\end{aligned}$$

by a combinatorial argument.

Bonus: Generalizes to Shishuo's "Glaisherized" version, where $2r \to r, r$ is replaced by $dr \to r^d$. Then the term involving parts is

$(\#$ parts in partitions with no multiples of $d-$

$\#$ parts in partitions no part repeated d or more times$)/(d-1)$.

Timeline

4/22/17 George Beck conjectures identity between part count difference and partitions with one even part (OEIS)

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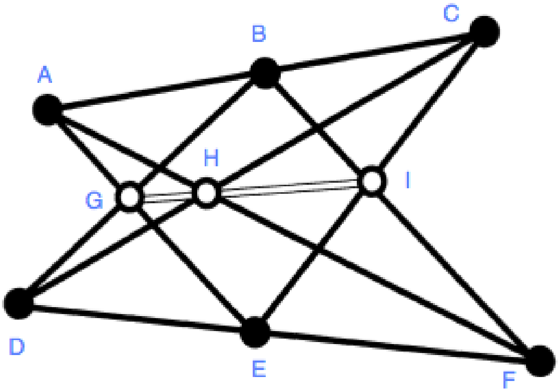
6/7/17 Bijection related to count differences presented here.

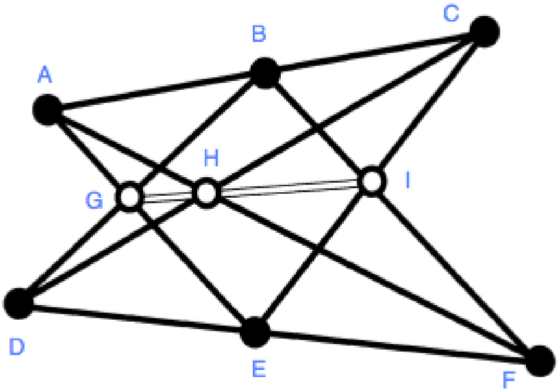
Break time!

Geometry result: Pappus of Alexandria, 4th Century CE



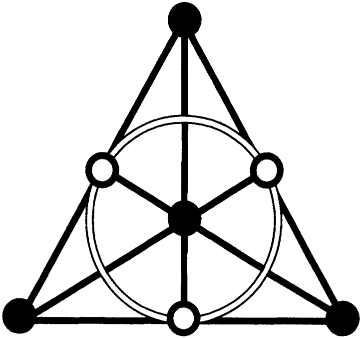
Configurations: Ancient

Geometry result: Pappus of Alexandria, 4th Century CE



9_3 configuration: 9 lines, 9 points, each line contains 3 points, each point is on 3 lines; each pair of points on at most 1 line.

Configurations: 19th Century



Fano plane, named for Gino Fano (1871–1952), a 7_3 configuration (combinatorial configuration, not requiring actual lines).

Configurations: 19th Century

- 1881 Kantor enumerates 1 8_3 configuration, 3 9_3, and 10 10_3 (with drawings, so geometric).
- 1887 Martinetti enumerates 31 11_3 configurations are develops recursive method to build $(n+1)_3$ configurations from an n_3.
- 1889 Schröter proved one of the 10_3 configurations cannot be realized with lines (so 9 geometric, 1 just combinatorial).
- 1895 Daublesky enumerates 228 12_3 configurations.

Many other results for more general configurations, rich "classical period"* ends with 1926 Steinitz book.

> ** characterized more by enthusiasm for configurations than by solid mathematical achievement.* —*Grünbaum*

It might be mentioned here that there was a time when the study of configurations was considered the most important brach of all geometry.

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The author would like to conjecture that this is the greatest exaggeration of the truth that can be found in any of Hilbert's writings.

—Branko Grünbaum, 2009, *Configurations of Points and Lines*, AMS Graduate Studies #103

Configurations: Recent

After "dark ages" (with exceptions: Levi, Dorwart, Coxeter), a renaissance.

- 1988 Sturmfels & White show that the 31 11_3 and the 228 12_3 configurations are geometric with rational points.
- 1990 Gropp finds a missing 229th 12_3 configuration (and shows the list is complete).
- 1999–2006 various computer enumerations of (combinatorial) n_3 configurations up to $n = 19$ (approximately 7.6×10^9).

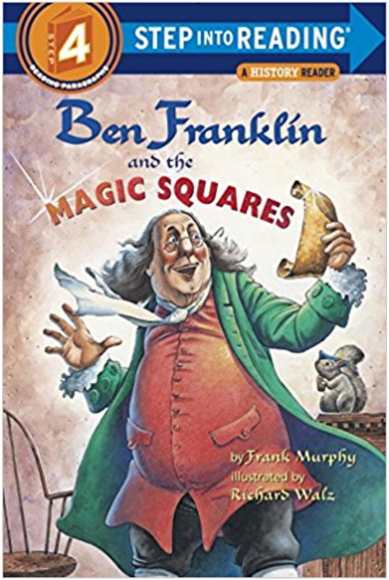
Currently 80+ recent MathSciNet entries on (these) configurations.

Albrecht Dürer, Melancholy, 1514

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Magic!


MELENCOLIA I


4
STEP INTO READING
A HISTORY READER
Ben Franklin
and the
MAGIC SQUARES
by Frank Murphy
illustrated by
Richard Walz

Matrix Approach

For what n can we label the points in an n_3 configuration such that the sums along each lines are the same?

Fano 7_3 configuration as table and matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

NB: Rows correspond to lines, columns to points. Not the standard adjacency or incidence matrix from graph theory.

Fano Magic?

Magic labeling for the unique 7_3 configuration?

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{pmatrix} = \begin{pmatrix} s \\ s \\ s \\ s \\ s \\ s \\ s \end{pmatrix}$$

has solution $a = b = \cdots = g = s/3$: trivial solution with label 1 on each point, "magic sum" 3.

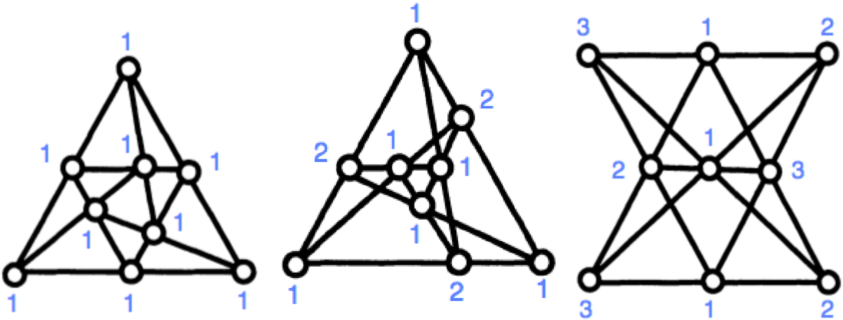
The unique 8_3 also has only the trivial all-equal labeling to make equal sums on all lines.

But of the 3 9_3 configurations, two of the corresponding incidence matrices are noninvertible.

$$(a, b, c, d, e, f, g, h, i) = (a, s-2a, a, a, a, a, s-2a, s-2a, a)$$
$$(a, b, c, d, e, f, g, h, i) = (a, b, s-a-b, s-a-b, b, a, a, b, s-a-b)$$

Labeled 9_3 configurations

Constant line sums with maximal number of distinct labels.



Call these 1-, 2-, and 3-partially magic, respectively.

Partially magic configurations

Table of n_3 counts by k-partially magic status.

	Σ	1	2	3	4	5	6	7	8	9	10	11	12
7	1	1											
8	1	1											
9	3	1	1	1									
10	10	8	0	2									
11	31	22	0	5	0	1	1	1	0	0	0	1	
12	229	135	29	26	11	11	4	5	2	3	1	1	1

Minimal Magic Configuration

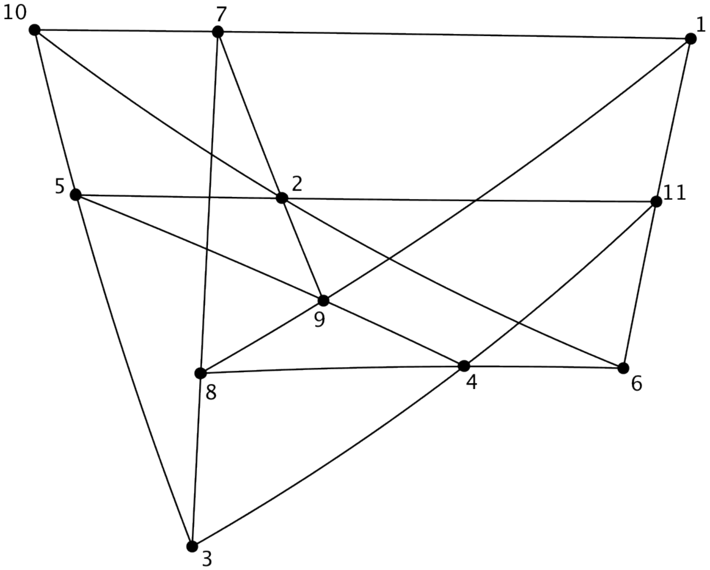


Table Approach

Rather than an exhaustive examination of noninvertible incidence matrices, work directly from the table. (Like KenKen.)

For labels $1, 2, \ldots, 11$, each appearing three times, the constant sum must be

$$\frac{3(1 + \cdots + 11)}{11} = \frac{3 \cdot 66}{11} = 18.$$

Note that no pair of $1, 2, 3$ can be in the same column (on the same line) since no 11 is the maximum label, could not make column sum of 18.

Table Approach

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & \bullet & \bullet \\ & & & & & & & & & & \end{pmatrix}$$

4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 8, 9, 9, 9, 10, 10, 10, 11, 11, 11

Table Approach

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & \bullet & \bullet \\ 6 & 7 & 8 & & & & & & & & \\ 11 & 10 & 9 & & & & & & & & \end{pmatrix}$$

4, 4, 4, 5, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 10, 11, 11

Table Approach

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & \bullet & \bullet \\ 6 & 7 & 8 & 5 & 6 & 7 & & & & & \\ 11 & 10 & 9 & 11 & 10 & 9 & & & & & \end{pmatrix}$$

4, 4, 4, 5, 5, 6, 7, 8, 8, 9, 10, 11

Table Approach

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \\ 6 & 7 & 8 & 5 & 6 & 7 & 4 & & & & \\ 11 & 10 & 9 & 11 & 10 & 9 & 11 & & & & \end{pmatrix}$$

5, 5, 6, 7, 8, 8, 9, 10

Table Approach

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \\ 6 & 7 & 8 & 5 & 6 & 7 & 4 & & & & \\ 11 & 10 & 9 & 11 & 10 & 9 & 11 & & & & \end{pmatrix}$$

$$\underline{5,\ 5},\ 6,\ 7,\ \underline{8,\ 8},\ 9,\ 10$$

Table Approach

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \\ 6 & 7 & 8 & 5 & 6 & 7 & 4 & 5 & & 5 & \\ 11 & 10 & 9 & 11 & 10 & 9 & 11 & & 8 & & 8 \end{pmatrix}$$

6, 7, 9, 10

Table Approach

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \\ 6 & 7 & 8 & 5 & 6 & 7 & 4 & 5 & 7 & 5 & 6 \\ 11 & 10 & 9 & 11 & 10 & 9 & 11 & 10 & 8 & 9 & 8 \end{pmatrix}$$

Building Magic Configurations

Given a magic n_3 configuration, need three columns with

- disjoint labels
- including $n(n+1)/2$ (i.e., $1/3$ of the line sum)
- with two pair of labels satisfying

$$a + b = c + d = n(n+1)/2$$

- and two pair of labels satisfying

$$f + g = h + i = 3n(n+1)/2.$$

E.g., $\begin{pmatrix} 1 \\ 7 \\ 10 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 11 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$ in the magic 11_3 configuration.

Building Magic Configurations

Replace these with 5 columns and adjust the remainder to give $n+2$ columns each with sum $3(n+1)(n+2)/2$.

$$\begin{pmatrix}1\\7\\10\end{pmatrix},\begin{pmatrix}2\\5\\11\end{pmatrix},\begin{pmatrix}4\\6\\8\end{pmatrix}\Longrightarrow\begin{pmatrix}0\\ \\12\end{pmatrix},\begin{pmatrix}0\\ \\ \end{pmatrix},\begin{pmatrix}0\\ \\ \end{pmatrix},\begin{pmatrix} \\ \\12\end{pmatrix},\begin{pmatrix} \\ \\12\end{pmatrix}$$

Building Magic Configurations

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$$\begin{pmatrix}1\\7\\10\end{pmatrix}, \begin{pmatrix}2\\5\\11\end{pmatrix}, \begin{pmatrix}4\\6\\8\end{pmatrix} \Longrightarrow \begin{pmatrix}0\\6\\12\end{pmatrix}, \begin{pmatrix}0\\7\\11\end{pmatrix}, \begin{pmatrix}0\\8\\10\end{pmatrix}, \begin{pmatrix}1\\5\\12\end{pmatrix}, \begin{pmatrix}2\\4\\12\end{pmatrix}$$

Now raise each label here and in the remaining columns of the n_3 configuration by 1.

Building Magic Configurations

$$\begin{pmatrix} 1 & \widehat{1} & 1 & \widehat{2} & 2 & 2 & 3 & 3 & 3 & 4 & \widehat{4} \\ 6 & 7 & 8 & 5 & 6 & 7 & 4 & 5 & 7 & 5 & 6 \\ 11 & 10 & 9 & 11 & 10 & 9 & 11 & 10 & 8 & 9 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 7 \\ 10 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 11 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 \\ 6 \\ 12 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \\ 11 \end{pmatrix}, \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 12 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} \widehat{1} & \widehat{1} & \widehat{1} & \widehat{2} & 2 & 2 & \widehat{3} & 3 & 3 & 4 & 4 & 4 & 5 \\ 7 & 8 & 9 & 6 & 7 & 9 & 5 & 7 & 8 & 5 & 6 & 8 & 6 \\ 13 & 12 & 11 & 13 & 12 & 10 & 13 & 11 & 10 & 12 & 11 & 9 & 10 \end{pmatrix}$$

Magic 13_3 Configurations

$$\begin{pmatrix} \widehat{1} & \widehat{1} & \widehat{1} & \widehat{2} & 2 & 2 & \widehat{3} & 3 & 3 & 4 & 4 & 4 & 5 \\ 7 & 8 & 9 & 6 & 7 & 9 & 5 & 7 & 8 & 5 & 6 & 8 & 6 \\ 13 & 12 & 11 & 13 & 12 & 10 & 13 & 11 & 10 & 12 & 11 & 9 & 10 \end{pmatrix}$$

from the recursion, and also

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 \\ 7 & 8 & 9 & 6 & 8 & 9 & 5 & 6 & 7 & 5 & 7 & 8 & 6 \\ 13 & 12 & 11 & 13 & 11 & 10 & 13 & 12 & 11 & 12 & 10 & 9 & 10 \end{pmatrix}.$$

Continuing On

The two magic 13_3 configurations have, respectively, 4 and 5 triples from which magic 15_3 configurations can be built (although need to check for isomorphisms).

Magic n_3 configurations exist for all odd $n \geq 11$.

Show that at least one desired triple of lines exists, apply recursion.

Actually many more: n	11	13	15	17	19
# magic configurations	1	2	18	115	2338

Evens

Magic n_3 configurations with labels $1, 2, \ldots, n$ require n odd. E.g., not possible to use labels $1, 2, \ldots, 12$ since

$$\frac{3(1 + \cdots + 12)}{12} = \frac{39}{2};$$

can find one with sum 20 using the label 13 rather than 11:

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 5 \\ 6 & 7 & 9 & 5 & 6 & 8 & 4 & 5 & 8 & 6 & 7 & 7 \\ 13 & 12 & 10 & 13 & 12 & 10 & 13 & 12 & 9 & 10 & 9 & 8 \end{pmatrix}.$$

One can make such "weakly magic" labelings (of this same configuration) with sums 20, 22, and all greater integers (can make sum $s + 3$ from s, just need to exhibit 20, 22, and 24).

Evens

Know from the matrix approach that 12 is the least possible even with weakly magic n_3 (can also show by trying to produce table).

Trickier since minimal sum $s > 3n(n+1)/2$ varies for even n:

n	12	14	16	18
minimum sum	20	24	27	29

Possible to modify the recursion based on a suitable triple? Not clear what would play the role of $n(n+1)/2$. Also, could not have the sum of 20 for the 12_3 go up by 3 for a 14_3 configuration.

To-do list & open questions

- Prove, for each odd n, that some magic configuration has a triple required for the recursion.
- Understand more about even n.
- Explore partially magic notion: Way to extend without needing the configuration matrix? Geometric meaning?

	Σ	1	2	3	4	5	6	7	8	9	10	11	12
9	3	1	1	1									
10	10	8	0	2									
11	31	22	0	5	0	1	1	1	0	0	0	1	
12	229	135	29	26	11	11	4	5	2	3	1	1	1
13	2036												