Hands-On Combinatorics: Pedagogy & Colorful Proofs

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Mathematics Seminar
Mahidol University International College
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Talk ingredients

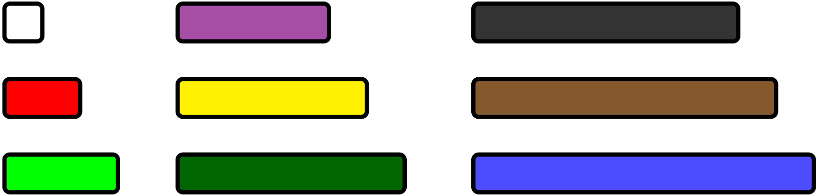
This presentation relates to

- integer compositions
- integer sequences & triangles
- an elementary school teaching aid
- K-12 teacher professional development
- undergraduate combinatorics
- book project

A mix of pedagogy and combinatorial proofs with historical notes.

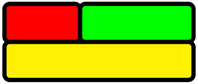
Cuisenaire rods

Georges Cuisenaire of Thium, Belgium (current pop. ∼14,000), developed in the early 1930s to help his elementary school students better visualize numbers. Wooden rods color-coded by length:



Cuisenaire rods

For instance, to see that $2 + 3 = 5$,



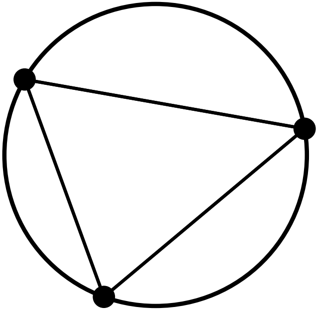
- Almost unknown beyond his village until 1951, then adapted across Belgium and he wrote up his methods (in French).
- University of London mathematics educator Caleb Gattegno became a big advocate, 1953 book *Numbers in Colour*.
- Popular around the world in elementary education 1950s - 1970s, waned with "new math" backlash.

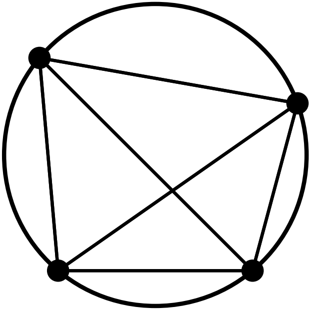
Transition to combinatorics

- First connection to higher math made in 1992 article “Cuisenaire Rods Go To College” by Phyllis Chen and collaborators, published in *PRIMUS* (Problems, Resources, and Issues in Mathematics Undergraduate Studies).
- I developed a more extensive program (8 sessions, two hours each) used with teachers in New Jersey, Colorado, Oregon, and adapted for students at Saint Peter’s University.
- I’m finishing up a book to be published by the American Mathematical Society.

Preliminary problem

Connect n points in general position on a circle with all possible chords. How many regions are created inside the circle?





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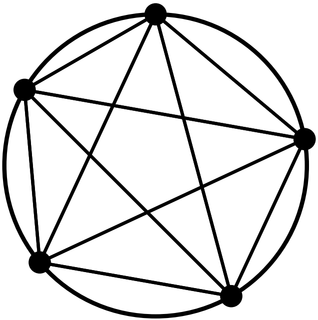

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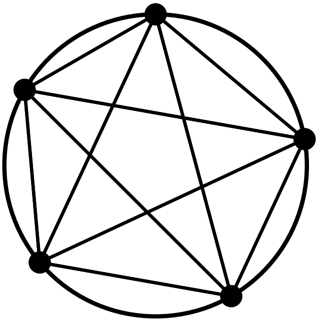
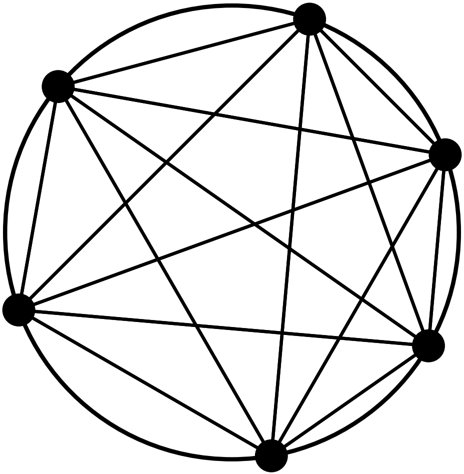
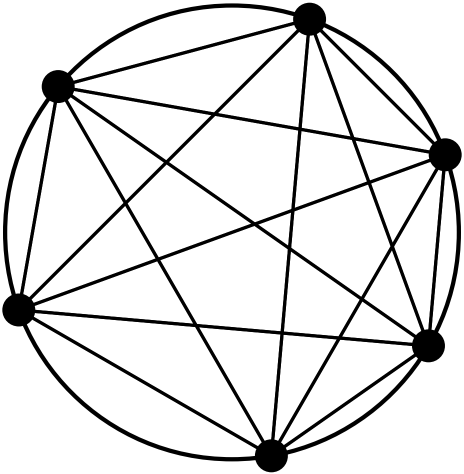


points	1	2	3	4	5
regions	1	2	4	8	16

Preliminary problem



Preliminary problem



But 6 points gives 31 regions!

Preliminary problem

From Leo Moser's "On the dangers of induction" in *Mathematics Magazine* 23 (1949).

points	1	2	3	4	5	6	7	8	9	10
regions	1	2	4	8	16	31	57	99	163	256

There is a pattern (revealed at the end), but perhaps not your first guess.

Presented as a cautionary tale, motivation for needing to prove "obvious" patterns.

Trains 1

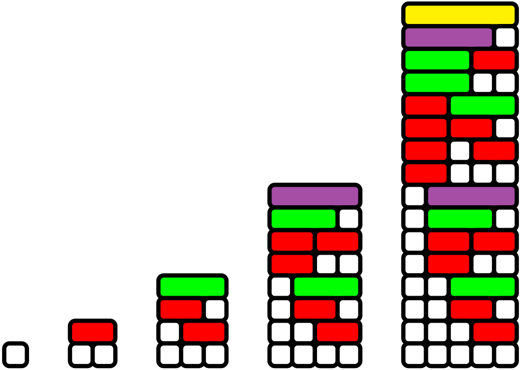
Call an individual Cuisenaire rod a car, a sequence of cars makes a train. Some length 7 trains:

How many length *n* trains are there?

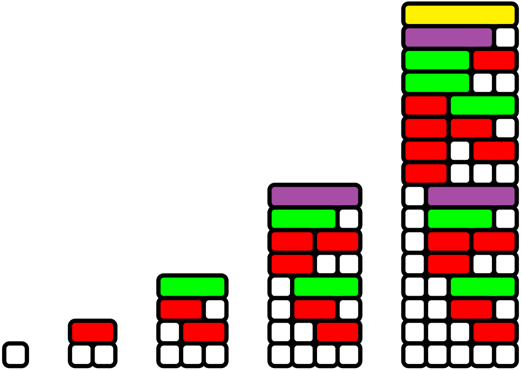
Frequent early issue: Among length 3 trains, are the following distinct?

Pedagogical approach: Let them discuss but eventually have everyone use the same approach. Not about correct or incorrect answer, rather adopting a consistent convention.

Trains 1



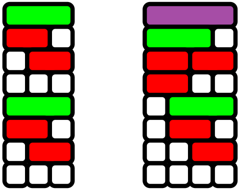
Trains 1



length	1	2	3	4	5
trains	1	2	4	8	16

From Moser's problem, they are suspicious of the pattern. Here the pattern is in fact powers of 2, but how can they be sure?

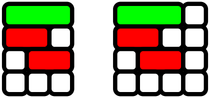
Given 2 sets of length 3 trains, can you build the length 4 trains?



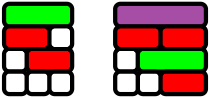
Common approach: Add a white car on the left on one set of length 3 trains, add a white car on the right of the other set. Messy *ad hoc* approaches to deal with duplicates & missing trains.

Trains 1

Add a white car at the end of one set.



Look for a simple transformation connecting the other set of length 3 trains and the remaining length 4 trains.



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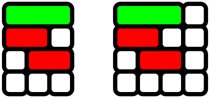

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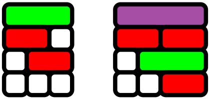


Extend the last car by one. (Using a consistent order for the trains can help suggest combinatorial arguments.)

Trains 1

Number of length n trains: $t(n) = 2^{n-1}$.

First proof: The "add one, extend last car" argument shows $t(n) = 2t(n-1)$. Initial value $t(1) = 2^0 = 1$.

De-emphasis on the rigors of one-to-one and onto. Some discussion of not generating the same train in different ways and how to "go backwards."

With K-8 teachers who may not be as comfortable with algebra and functional notation, we work without $t(n)$ and similar notation. Instead, "there are 2^{n-1} length n trains" — emphasis on combinatorial thinking rather than symbolic proficiency.

Trains 1

"The expression 2^{n-1} suggests $n-1$ binary choices. Can we explain the number of length n trains more directly?"

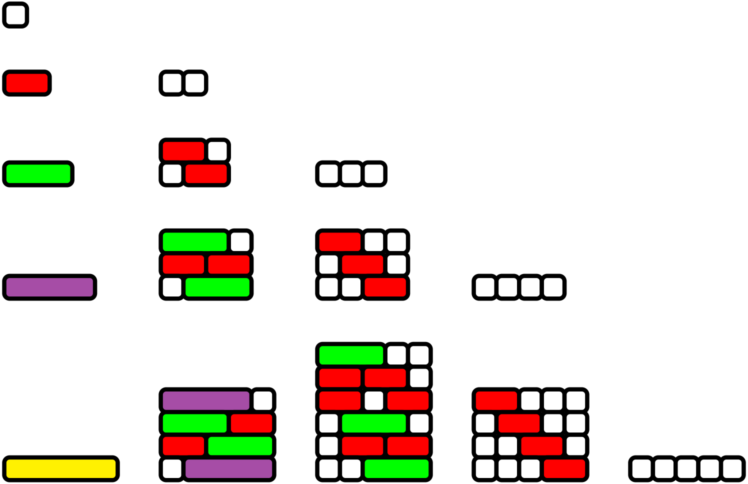
Number of length n trains: $t(n) = 2^{n-1}$.

Second proof: A length n train is determined by its length $n-1$ cut-join sequence.

The tools of adding cars, extending cars, and the cut-join sequence are enough for everything in the course.

Trains 2

Let $t(n,k)$ be the number of length n trains made of k cars.



Trains 2

Number of length n trains with k cars: $t(n,k) = \binom{n-1}{k-1}$.

Proof: A train with k cars has $k-1$ cuts in its cut-join sequence. Number of ways to choose $k-1$ out of $n-1$ positions is $\binom{n-1}{k-1}$.

Many teachers (and students!) have a shaky understanding of Pascal's triangle, most just remember the algorithm for computing an entry from the ones above it. We define Pascal's triangle in terms of trains, so Pascal's lemma needs to be proved.

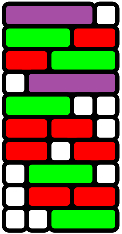
But one identity comes for free:

Pascal's triangle row sums: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof: $\sum_{k=0}^{n} \binom{n}{k} = \sum_{j=1}^{n+1} t(n+1,j)$ just re-combines all length $n+1$ trains that had been partitioned by number of cars. So $\sum_{k=0}^{n} \binom{n}{k} = t(n+1) = 2^n$.

Trains 2

Trains for $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$, i.e., $t(5,2) + t(5,3) = t(6,3)$:



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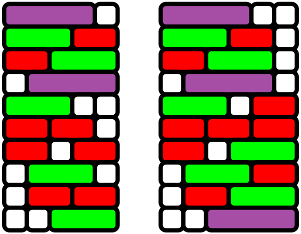


Pascal's lemma: $t(n,k) = t(n-1,k) + t(n-1,k-1)$.

Proof: For each train in $T(n-1,k-1)$, add a white car at the end (one longer, one more car). For each train in $T(n-1,k)$, extend the last car by one (one longer, same number of cars).

Trains 2

The "hockey stick" result:

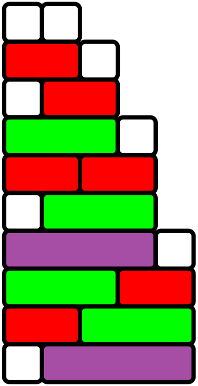
1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

(Vertical) hockey stick identity:

For positive k and ℓ, $$\sum_{m=0}^{\ell-1} t(k+m,k) = t(k+\ell,k+1).$$

The "hockey stick" combinatorial proof idea:

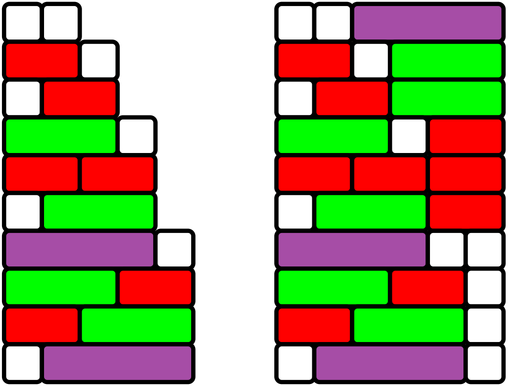
$$t(2,2) + t(3,2) + t(4,2) + t(5,2) = t(6,3).$$



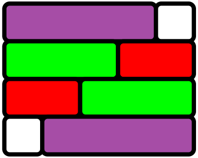
Trains 2

The "hockey stick" combinatorial proof idea:

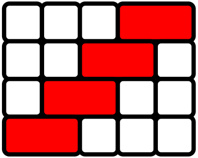
$$t(2,2) + t(3,2) + t(4,2) + t(5,2) = t(6,3).$$



Each row of Pascal's triangle is a palindrome, i.e., reads the same left-to-right as right-to-left. E.g., 1 3 3 1 and 1 4 6 4 1.



J J J C	C C C J
J J C J	C C J C
J C J J	C J C C
C J J J	J C C C

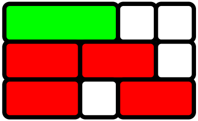


Symmetry by number of cars: $t(n,k) = t(n, n+1-k)$.

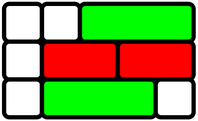
Proof notes: Swaps cuts and joins. A k car train has $k-1$ cuts among its length $n-1$ cut-join sequence. Its complement has $(n-1)-(k-1) = n-k$ cuts, so $n-k+1$ cars.

Trains 2

What if we take complements of the trains "in the middle," e.g., the ones counted by $\binom{4}{2} = 6$?



J J C C	C C J J
J C J C	C J C J
J C C J	C J J C



Central binomial coefficients are even.

Given $\mathrm{T} \in T(2k-1, k)$, since $(2k-1)+1-k = k$, the complement $\overline{\mathrm{T}} \in T(2k-1, k)$. The operation is an involution with no fixed points, so these trains come in pairs.

Trains 2

Alternating row sums in Pascal's triangle:

$$\begin{array}{ccccccccccc} 1 & - & 1 & & & & & & & = & 0 \\ 1 & - & 2 & + & 1 & & & & & = & 0 \\ 1 & - & 3 & + & 3 & - & 1 & & & = & 0 \\ 1 & - & 4 & + & 6 & - & 4 & + & 1 & = & 0 \end{array}$$

Every other row is obvious from symmetry, but why should $1 - 4 + 6 - 4 + 1 = 0$?

Trains 2

Alternating row sums in Pascal's triangle:

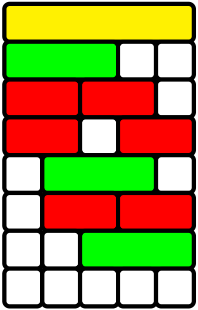
$$\begin{array}{ccccccccccccc} 1 & - & 1 & & & & & & & & & = & 0 \\ 1 & - & 2 & + & 1 & & & & & & & = & 0 \\ 1 & - & 3 & + & 3 & - & 1 & & & & & = & 0 \\ 1 & - & 4 & + & 6 & - & 4 & + & 1 & & & = & 0 \end{array}$$

Every other row is obvious from symmetry, but why should $1 - 4 + 6 - 4 + 1 = 0$?

The proof we don't want using the binomial theorem:

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k}.$$

Prove instead that $1 + 6 + 1 = 4 + 4$:



J J J J	J J J C
J J C C	J J C J
J C J C	J C J J
J C C J	J C C C
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C J C J	C J C C
C C J J	C C J C
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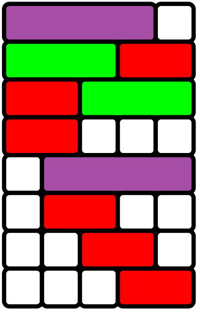


Switching the last cut-join changes between trains with an odd number of cars and trains with an even number of cars.

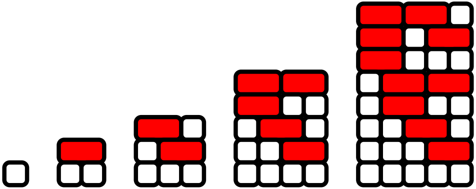
Trains 3

The real fun starts with placing restrictions on the compositions / trains. (Re. symbolic dynamics.) To start, consider three types:

- Compositions with only 1s and 2s / trains with only white and red cars. Write $t_{12}(n)$ for the number of such length n trains.
- Compositions with only odd parts / trains with only odd length cars. Notation $t_{\mathrm{odd}}(n)$.
- Compositions with no 1s / trains with no white cars. Notation $t_{\hat{1}}(n)$ ("hat" for omission).

Trains 3

Trains with only white and red cars, $t_{12}(n)$.



Trains 3

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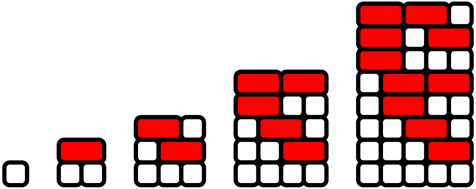
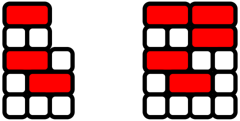
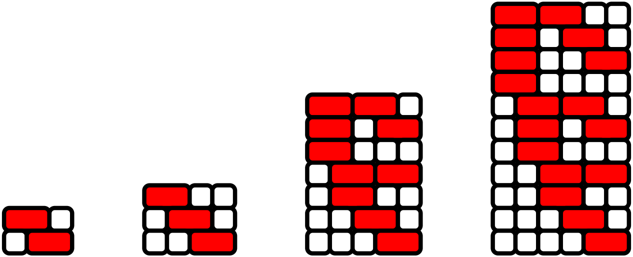


Fibonacci: Add white to length $n-1$, add red to length $n-2$.



Trains 3

One time, participants understood "made from white and red cars" to mean that both colors were required.

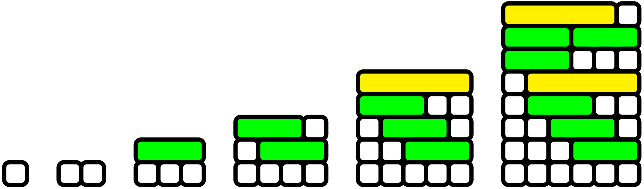


Sequence $2, 3, 7, 11, 20, 32, 54, 87$ (OEIS A245738), within 1 or 2 of a Fibonacci number, has its own degree four recurrence.

Balance between letting students explore and being on intended track.

Trains 3

Trains with only odd length cars, $t_{\text{odd}}(n)$.



Trains 3

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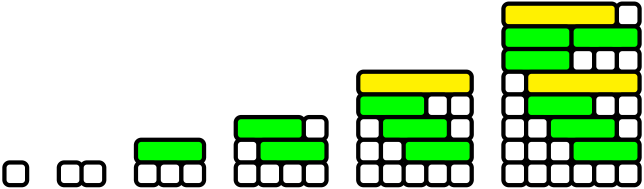
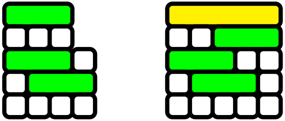
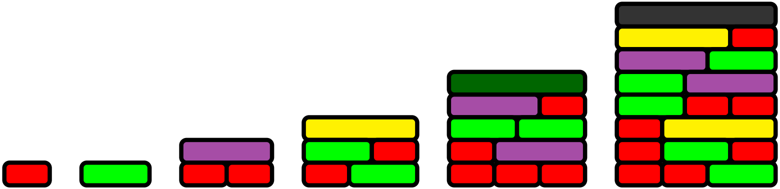


Add white to length $n-1$, extend last car of length $n-2$ by 2.



Trains 3

Trains with no white cars, $t_{\hat{1}}(n)$.



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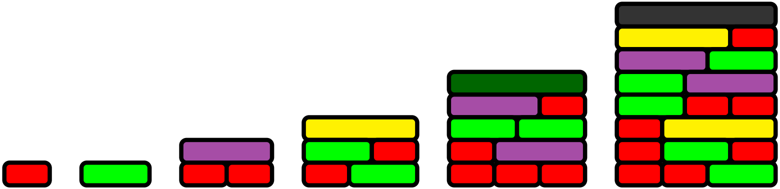
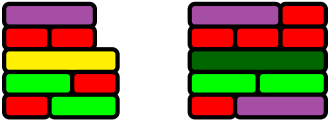


Extend last car of length $n-1$ by 1, add red to length $n-2$.

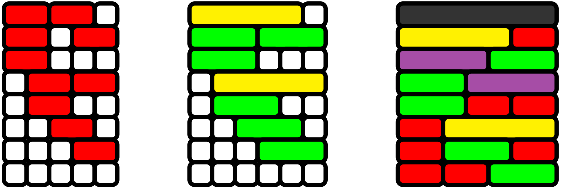


- $t_{12}(n)$ from counting patterns in Sanskrit poetry, long vowels twice as long as short vowels. E.g., long-long-short $\sim$ 221 and short-short-long-short $\sim$ 1121 both take 5 beats. Scholars involved in this include Piṅgala 5c. BCE, Virahāṅka 7c. CE, then Gopāla and Hemacandra both closer to but before Leonardo Pisano / Fibonacci.
- $t_{\text{odd}}(n)$ from an appendix in Augustus de Morgan's *Elements of Arithmetic*, fifth edition of 1846. "Required the number of ways in which a number can be compounded of odd numbers, different orders counting as different ways."
- $t_{\hat{1}}(n)$ from an 1876 *Messenger of Mathematics* research note by Arthur Cayley, "u_n = number of partitions of n, no part less than 2, order attended to ..."

Trains 3

Fibonacci $f(1) = f(2) = 1$, $f(n) = f(n-1) + f(n-2)$ for $n \geq 2$.

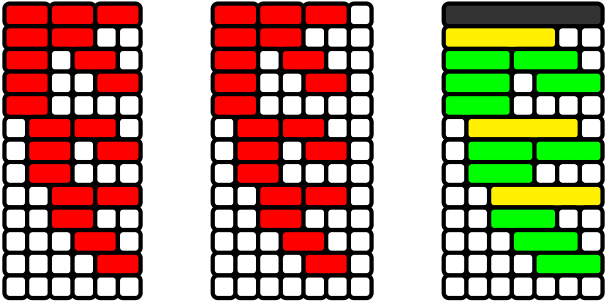
$$t_{12}(n-1) = t_{\text{odd}}(n) = t_{\hat{1}}(n+1) = f(n).$$



Direct connection between these various "Fibonacci trains"?

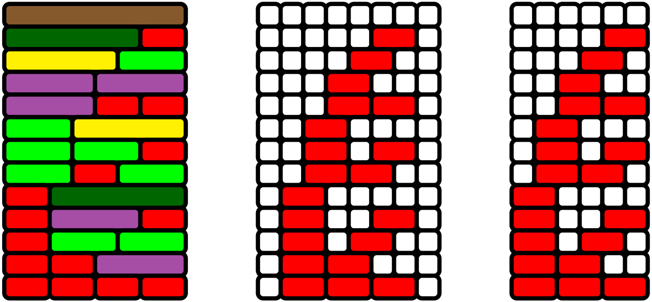
Trains 3

$$T_{12}(n) \cong T_{\text{odd}}(n+1)$$



Add a white, (k reds followed by a white) $\mapsto$ $(2k+1)$.

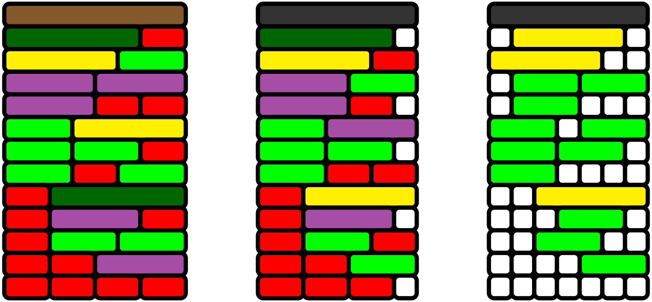
$$T_{\hat{1}}(n) \cong T_{12}(n-2)$$



Complementary train, remove whites at beginning & end.

Trains 3

$$T_{\hat{1}}(n) \cong T_{\text{odd}}(n-1)$$



Decrease last part by 1, any $(2k) \mapsto (1, 2k-1)$.

Trains 8

To prove a Fibonacci identity combinatorially, one of the "Fibonacci compositions" may be more conducive than the others.

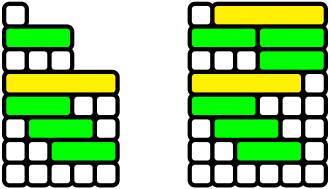
$$f(1) + f(3) + \cdots + f(2n-1) = f(2n)$$

Trains 8

To prove a Fibonacci identity combinatorially, one of the "Fibonacci compositions" may be more conducive than the others.

$$f(1) + f(3) + \cdots + f(2n-1) = f(2n)$$

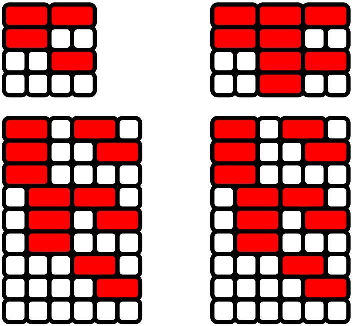
$$t_{\text{odd}}(1) + t_{\text{odd}}(3) + \cdots + t_{\text{odd}}(2n-1) = t_{\text{odd}}(2n)$$



Modify a length $(2k-1)$ train by adding $(2n-2k+1)$.

$$(f(n))^2 + (f(n+1))^2 = f(2n+1)$$

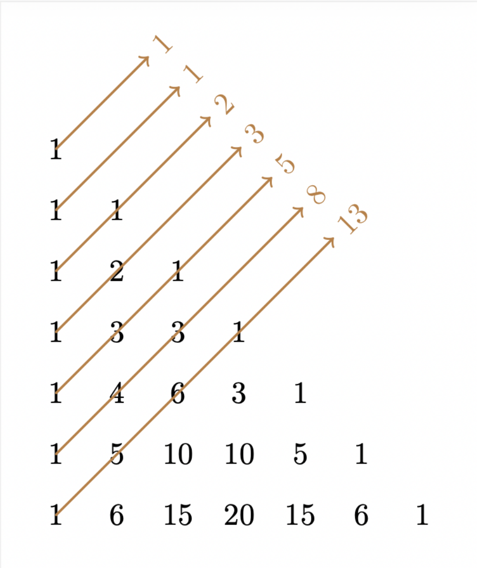
$$(t_{12}(n-1))^2 + (t_{12}(n))^2 = t_{12}(2n)$$



Squares by concatenation. Put red in the middle of shorter trains.

Trains 8

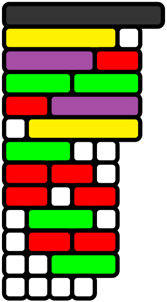
Fibonacci numbers in Pascal's Triangle



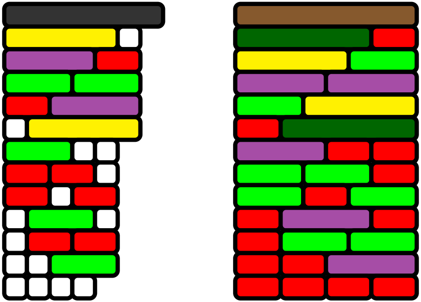
Trains 8

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = f(n+1)$$

$$t(n+1,1) + t(n,2) + t(n-1,3) + \cdots = ?$$



$$t(7,1) + t(6,2) + t(5,3) + t(4,4) = t_{\hat{1}}(8)$$



Trains that are k too short have k cars: Expand each car by 1.

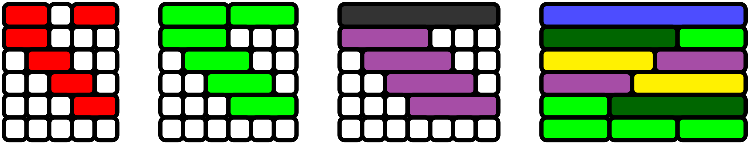
Trains 5

A fleet of restricted trains:

a. Trains made with only white and green cars.
b. Trains made with only red and green cars.
c. Trains made with only white and red cars having no adjacent white cars.
d. Trains made with only white and red cars having no adjacent red cars.
e. Trains made with no white or red cars.
g. Trains made with only odd length cars but no white cars.
h. Trains made with only cars whose lengths are 1 modulo 3.
i. Trains made with only cars whose lengths are 2 modulo 3.

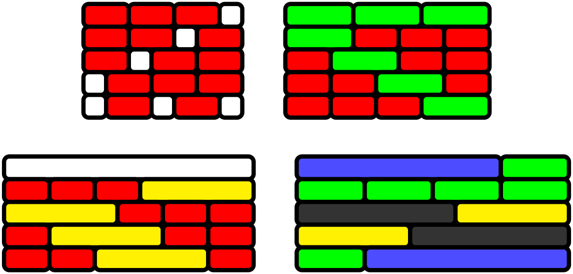
Trains 5

Four are counted by Narayana's cows sequence, 1, 2, 3, 4, 6, 9, 13, 19, with recurrence $y(n) = y(n-1) + y(n-3)$.



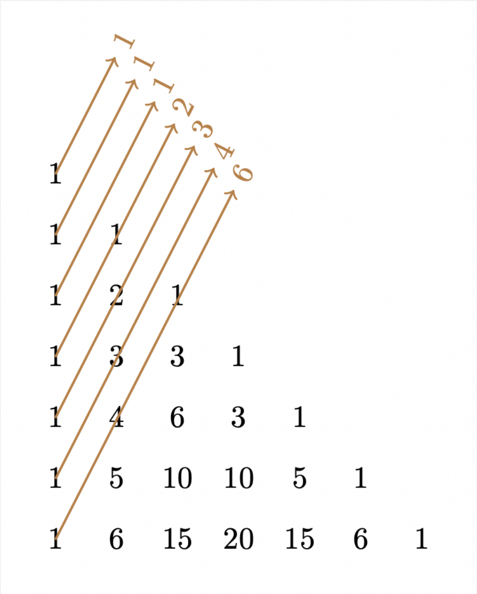
Proofs showing each type satisfies the recurrence, bijections connecting all four types.

The other four are counted by the Padovan sequence, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, with recurrence $v(n) = v(n-2) + v(n-3)$.

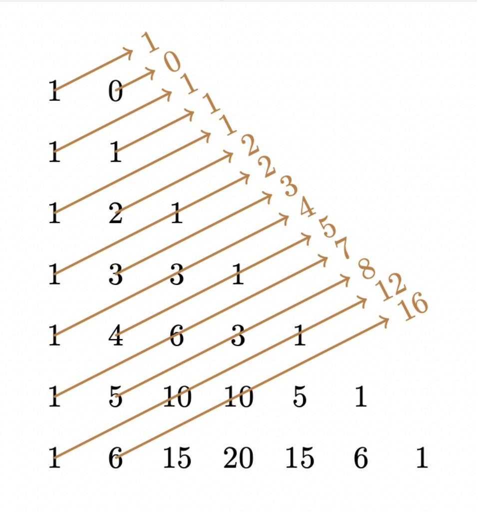


Analogous proofs and bijections.

Narayana's cows numbers in Pascal's triangle

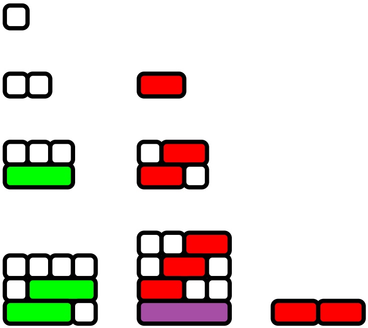


Padovan numbers in Pascal's triangle



Trains 7

Organize all trains by number of even length cars.



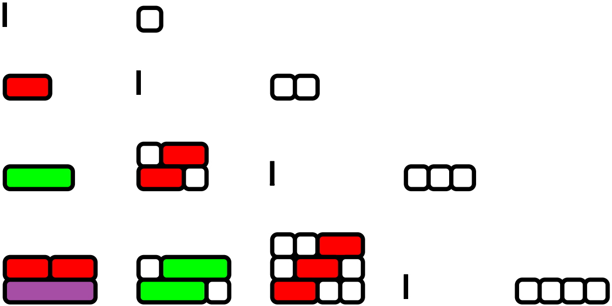
First column is all odd parts, Fibonacci.

1			
1	1		
2	2		
3	4	1	
5	8	3	
8	15	8	1

$$t_e(n,k) = t_e(n-1,k) + t_e(n-2,k) + t_e(n-2,k-1).$$

Row sums 2^{n-1}, diagonal sums Pell numbers, down diagonal sums a tribonacci sequence, alternating row sum 0, . . . A119473

Organize all trains by number of white cars.



First column is no 1s, Fibonacci.

Trains 7

0	1					
1	0	1				
1	2	0	1			
2	2	3	0	1		
3	5	3	4	0	1	
5	8	9	4	5	0	1

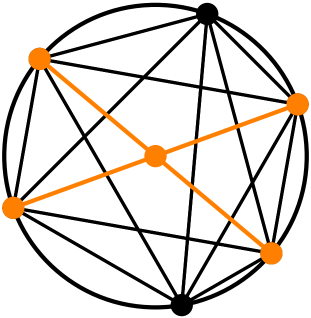
$$t_w(n,k) = t_w(n-1,k) + t_w(n-1,k-1) + t_w(n-2,k) - t_w(n-2,k-1)$$

Trains 7

0	1					
1	0	1				
1	2	0	1			
2	2	3	0	1		
3	5	3	4	0	1	
5	8	9	4	5	0	1

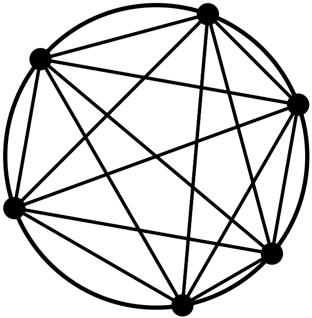
Row sums 2^{n-1}, diagonal sums nice (no 1s, two types of 2s).
Student in final project found a nice hockey stick variant. A105422

Solve Moser's problem about counting circular regions determined by chords using Euler's formula for planar graphs (polyhedra).



$$V = n + \binom{n}{4}$$

Trains 4



$$E = \frac{1}{2}\left(n(n+1) + 4\binom{n}{4}\right)$$

Trains 4

From $V - E + F = 2$, really want $F - 1$ since we don't count the "outside" face.

$$\begin{aligned} F - 1 &= E - V + 1 \\ &= \frac{n(n+1)}{2} + 2\binom{n}{4} - \left(n + \binom{n}{4}\right) + 1 \\ &= 1 + (n-1) + \frac{(n-1)(n-2)}{2} + \binom{n}{4} \\ &= \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4}, \end{aligned}$$

i.e., the first five entries of the $(n-1)$st row of Pascal's triangle.

Trains 4

n	$\binom{n-1}{0}$	$\binom{n-1}{1}$	$\binom{n-1}{2}$	$\binom{n-1}{3}$	$\binom{n-1}{4}$	sum
1	1	0	0	0	0	1
2	1	1	0	0	0	2
3	1	2	1	0	0	4
4	1	3	3	1	0	8
5	1	4	6	4	1	16
6	1	5	10	10	5	31
7	1	6	15	20	15	57
8	1	7	21	35	35	88
9	1	8	28	56	70	163
10	1	9	36	84	126	256