Equality versus Greed in Two Person Division of Indivisible Items

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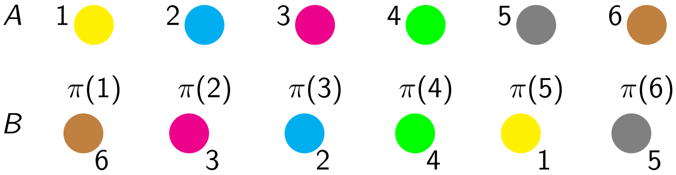
Set Up

Players A and B are to split $2n$ indivisible items, each receiving n of them. Each player ranks their preferences over the $2n$ items.

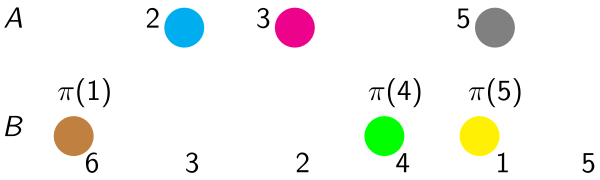


Number the items according to A's preferences; all information is then encoded in B's preferences, which can be considered as a permutation in S_{2n}, here $\pi = (6, 3, 2, 4, 1, 5) \in S_6$.

Scoring

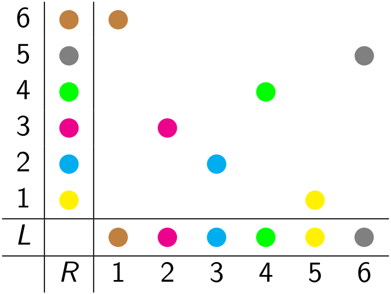
Utility theory: How do we quantify each player's outcome?

A player's score under an allocation is the rank sum of the n items received *with respect to her preferences*.



Because players can have different preferences, both can simultaneously do better than you might think. Here, A gets $\{2, 3, 5\}$ for rank sum 10 and B gets $\{1, 4, 6\} = \{\pi(1), \pi(4), \pi(5)\}$ also for rank sum 10.

Best possible from $2n$ items is $1 + 2 + \cdots + n = (n^2 + n)/2$. Worst possible is $(n + 1) + \cdots + 2n = (2n^2 + n)/2$, although this almost never occurs in practice.



Permutation diagrams are one helpful algebraic combinatorics tool.

Some previous work

Most work on this considers alternating selection $ABAB\ldots$ with each player trying to optimize their outcome.

Knowledge states: Do players know each other's preferences? How can this information be used?

Four possibilities:

- Closed information: neither player knows the other's preferences.
- One-way information: exactly one player knows the other's preferences
- Open information: both players know the other's preferences.

Algorithms

For each knowledge state, we know an algorithm leading to the optimal result.

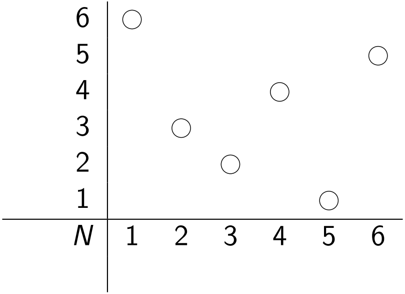
- Closed information: each player chooses their favorite item available at her turn. Naïve vs. Naïve, "top down."
- One-way information: The player with knowledge uses it to strategically delay preferred items to later turns. Algorithms proved optimal in H. & Jones 2009.
- Open information: Players work backwards from the end with what they will be "stuck with." Strategic vs. Strategic, "bottom up," shown optimal by Kohler & Chandrasekaharan 1971.

Algorithms: Naïve versus Naïve

Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A: $1, 2, 3, 4, 5, 6$
B: $6, 3, 2, 4, 1, 5$

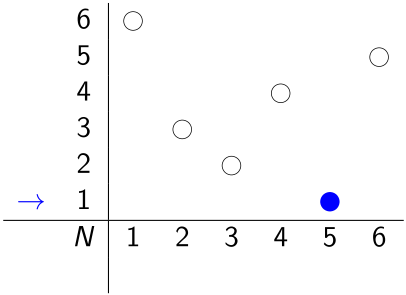
A	B



Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A: $1, 2, 3, 4, 5, 6$
B: $6, 3, 2, 4, 1, 5$

A	B
1	

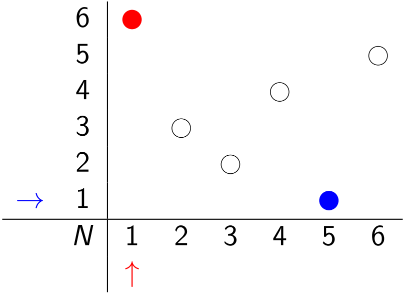


Algorithms: Naïve versus Naïve

Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A: $\quad 2, 3, 4, 5, 6$
B: $6, 3, 2, 4, \quad 5$

A	B
1	$6 = \pi(1)$

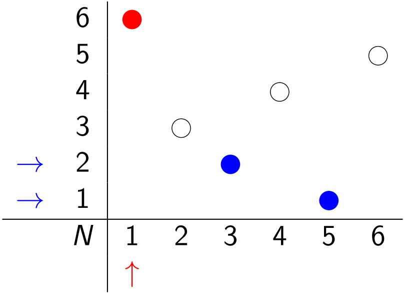


Algorithms: Naïve versus Naïve

Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A: $2, 3, 4, 5$
B: $3, 2, 4, \quad 5$

A	B
1	$6 = \pi(1)$
2	



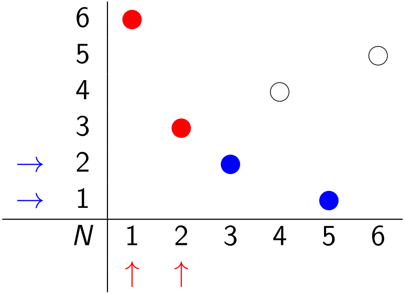
Algorithms: Naïve versus Naïve

Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A: $3, 4, 5$

B: $3, \quad 4, \quad 5$

A	B
1	$6 = \pi(1)$
2	$3 = \pi(2)$

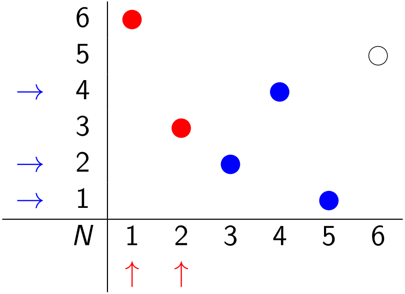


Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A: 4, 5

B: 4, 5

A	B
1	$6 = \pi(1)$
2	$3 = \pi(2)$
4	

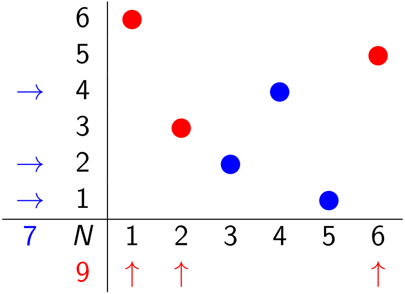


Naïve versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A: 5

B: 5

A	B
1	$6 = \pi(1)$
2	$3 = \pi(2)$
4	$5 = \pi(6)$
7	9



Strategic versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A knows π but B does not know A's order. In particular, A knows that B will not take 1 in the first two moves, so it is "safe" and A can focus on 3 and 2 first.

A: $1, 2, 3, 4, 5, 6$
B: $6, 3, 2, 4, 1, 5$

A	B
3	

Strategic versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A knows π but B does not know A's order. In particular, A knows that B will not take 1 in the first two moves, so it is "safe" and A can focus on 3 and 2 first.

A:

B: $6, \quad 2, 4, 1, 5$

A	B
3	$6 = \pi(1)$

Algorithms: Strategic versus Naïve

Strategic versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A knows π but B does not know A's order. In particular, A knows that B will not take 1 in the first two moves, so it is "safe" and A can focus on 3 and 2 first.

A: $1, 2, \quad 4, 5$
B: $\quad 2, 4, 1, 5$

A	B
3	$6 = \pi(1)$
2	

Strategic versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A knows π but B does not know A's order. In particular, A knows that B will not take 1 in the first two moves, so it is "safe" and A can focus on 3 and 2 first.

A:

B: $4, 1, 5$

A	B
3	$6 = \pi(1)$
2	$4 = \pi(4)$

Strategic versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A knows π but B does not know A's order. In particular, A knows that B will not take 1 in the first two moves, so it is "safe" and A can focus on 3 and 2 first.

A: $1,\quad 5$
B: $1, 5$

A	B
3	$6 = \pi(1)$
2	$4 = \pi(4)$
1	

Strategic versus Naïve for $\pi = (6, 3, 2, 4, 1, 5)$.

A knows π but B does not know A's order. In particular, A knows that B will not take 1 in the first two moves, so it is "safe" and A can focus on 3 and 2 first.

A:
B: 5

A	B
3	$6 = \pi(1)$
2	$4 = \pi(4)$
1	$5 = \pi(6)$
6	11

Algorithms: Naïve versus Strategic

Naïve versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

B does knows As order but A does not know π. In particular, B knows that A will not take $6 = \pi(1)$ ever, so it is "safe" and B can focus on $3 = \pi(2)$ first.

A: $1, 2, 3, 4, 5, 6$
B:

A	B
1	

Naïve versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

B does knows As order but A does not know π. In particular, B knows that A will not take $6 = \pi(1)$ ever, so it is "safe" and B can focus on $3 = \pi(2)$ first.

A: $, 2, 3, 4, 5, 6$

B: $6, 3, 2, 4, , 5$

A	B
1	$3 = \pi(2)$

Naïve versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

B does knows As order but A does not know π. In particular, B knows that A will not take $6 = \pi(1)$ ever, so it is "safe" and B can focus on $3 = \pi(2)$ first.

A: 2, 4, 5, 6

B:

A	B
1	$3 = \pi(2)$
2	

Naïve versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

B does knows As order but A does not know π. In particular, B knows that A will not take $6 = \pi(1)$ ever, so it is “safe” and B can focus on $3 = \pi(2)$ first.

A: 4, 5, 6

B: 6, 4, 5

A	B
1	$3 = \pi(2)$
2	$4 = \pi(4)$

Naïve versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

B does knows As order but A does not know π. In particular, B knows that A will not take $6 = \pi(1)$ ever, so it is "safe" and B can focus on $3 = \pi(2)$ first.

A: $5, 6$

B:

A	B
1	$3 = \pi(2)$
2	$4 = \pi(4)$
5	

Naïve versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

B does knows As order but A does not know π. In particular, B knows that A will not take $6 = \pi(1)$ ever, so it is "safe" and B can focus on $3 = \pi(2)$ first.

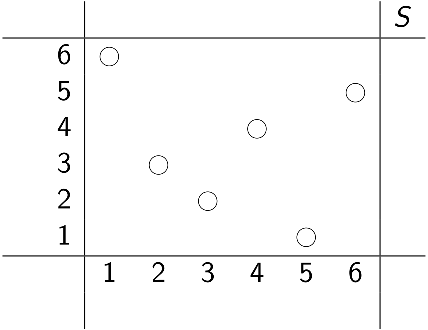
A: 6
B: 6

A	B
1	$3 = \pi(2)$
2	$4 = \pi(4)$
5	$6 = \pi(1)$
8	7

Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

A: $1, 2, 3, 4, 5, 6$
B: $6, 3, 2, 4, 1, 5$

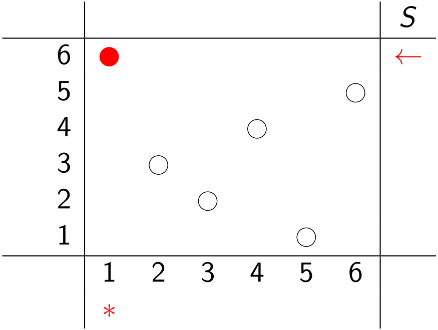
A	B



Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

A: $1, 2, 3, 4, 5, 6$
B: $6, 3, 2, 4, 1, 5$

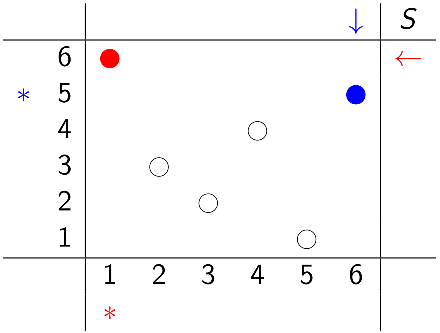
A	B
	$6 = \pi(1)$



Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

A: $1, 2, 3, 4, 5$
B: $\quad 3, 2, 4, 1, 5$

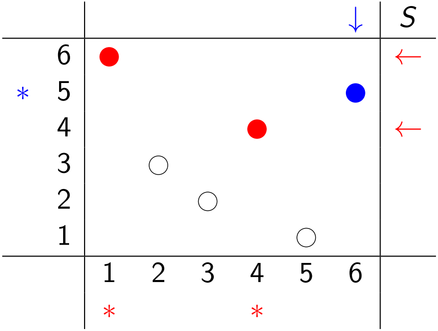
A	B
5	$6 = \pi(1)$



Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

A: $1, 2, 3, 4$
B: $3, 2, 4, 1$

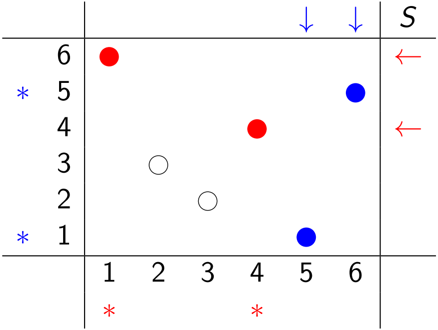
A	B
	$4 = \pi(4)$
5	$6 = \pi(1)$



Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

A: $1, 2, 3$
B: $3, 2, 1$

A	B
1	$4 = \pi(4)$
5	$6 = \pi(1)$

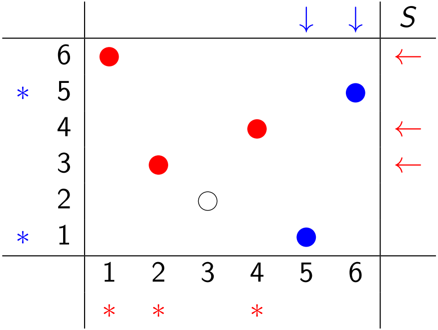


Algorithms: Strategic versus Strategic

Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

A: 2, 3
B: 3, 2

A	B
	$3 = \pi(2)$
1	$4 = \pi(4)$
5	$6 = \pi(1)$

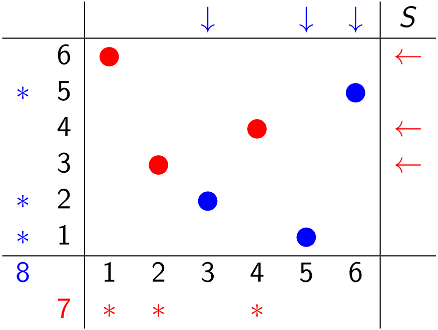


Algorithms: Strategic versus Strategic

Strategic versus Strategic for $\pi = (6, 3, 2, 4, 1, 5)$.

A: 1, 2, 3, 4, 5, 6
B: 6, 3, 2, 4, 1, 5

A	B
2	$3 = \pi(2)$
1	$4 = \pi(4)$
5	$6 = \pi(1)$
8	7



Some previous work

For alternating selection, $ABAB\ldots$.

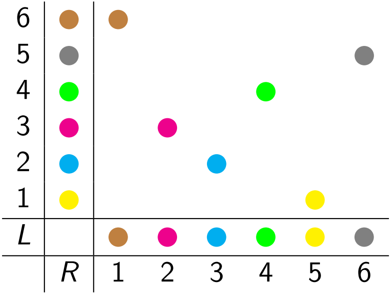
- (Brams & Taylor 1996) Algorithm with "contested pile"
- (H. & Jones 2009) Also, sequential Catalan number count possible outcomes for first & second players; connections to intervals in the S_{2n} left and right weak Bruhat partial orders
- (H., AMMCS 2013) NN and SS equal on average for both
- (Brams, Kilgour, Klamler 2014) AL algorithm (improves BT)

Also H., Keith, and Chelsea Grindatti (student) on fairest orders:

- ABBA
- ABABBA
- AABBBABA
- ABBAABABBA
- AABBBBAABAAB
- AAABBBBAABBBAA

Other motivations

At 7 sections meetings of the MAA:
“Symmetric Group and Fair Division: Does Knowledge Matter?”



Greed: Striving to minimize your rank sum
Spite: Greed using the other player’s preferences
Self-loathing: Greed using the reverse of your preferences
Altruism: Greed using the reverse of the other player’s preferences

Equality

Rather than greed, spite, etc., try equality as a motivation:

Allocate items so that both players' rank sums are equal.

- When can this be done?
- At what "cost"? I.e., how do average rank sums insisting on equality compare to those from other approaches?

For Comparison

Extreme results for alternating selection.

player	best	rank sum	worst	rank sum
A	$\{1,2,\ldots,n\}$	$\frac{n(n+1)}{2}$	$\{1,3,\ldots,2n-1\}$	n^2
B	$\pi(\{1,2,\ldots,n\})$	$\frac{n(n+1)}{2}$	$\pi(\{2,4,\ldots,2n\})$	$n(n+1)$

Averages over all preferences (equal for NN and SS).

$2n$	player	lowest	average	highest
6	A	6	7.00	9
	B	6	8.60	12
8	A	10	12.00	16
	B	10	14.18	20

One case considered before: How often do both players get their most preferred n items?

Example: $\pi = (6, 5, 7, 8, 2, 4, 3, 1)$

This happens whenever B's top choices are $n + 1, \ldots, 2n$ in whatever order (no matter what's known, the selection order, etc.). There are $(n!)^2$ such preferences, giving the small probability

$$\frac{(n!)^2}{(2n!)} = \frac{1}{\binom{2n}{n}}$$

that a randomly selected preference leads to this outcome.

Case $2n = 6$

For $(3!)^2 = 36$ preferences, both players can be allocated their top three items, rank sum 6 for both.

Rank sum 7? Possible when $\{\pi(1), \pi(2), \pi(4)\} \cap \{1, 2, 4\} = \varnothing$. Excluding the previously counted 36 preferences leaves 32 where the lowest possible equal rank sum division gives both players 7.

For $(3!)^2 = 36$ preferences, both players can be allocated their top three items, rank sum 6 for both.

Rank sum 7? Possible when $\{\pi(1), \pi(2), \pi(4)\} \cap \{1, 2, 4\} = \varnothing$. Excluding the previously counted 36 preferences leaves 32 where the lowest possible equal rank sum division gives both players 7.

Rank sum 8 can happen two ways: $1 + 2 + 5 = 1 + 3 + 4 = 8$. So

$$\pi(\{1,2,5\}) \cap \{1,2,5\} = \varnothing \quad \text{or} \quad \pi(\{1,2,5\}) \cap \{1,3,4\} = \varnothing \quad \text{or}$$
$$\pi(\{1,3,4\}) \cap \{1,2,5\} = \varnothing \quad \text{or} \quad \pi(\{1,3,4\}) \cap \{1,3,4\} = \varnothing$$

which, excluding the previous, describes 111 preferences.

Case $2n = 6$

rank sum	6	7	8	9	10
triples	123	124	125, 134	135, 126, 234	136, 145, 235
count	36	32	111	179	113

(Constrained partition problem for, here, all distinct 3-part partitions of k with parts at most 6—known, easy generating function.)

Reduces worst possible for second player from 12 to 10. Yet the weighted average is 8.64 (compare 7.00 for the first player in alternating selection, 8.60 for the second).

Also, $36 + 32 + 111 + 179 + 113 = 471$, only 65.4% of the 720.

Case $2n = 8$

10	11	12	13	14	15	16	17	18
576	540	2020	4055	8683	6593	7911	4003	1113

Note: The preference $(1, 2, 3, 4, 5, 6, 7, 8)$ is the most problematic in alternating selection: gives both players their worst possible outcomes, 16 for A and 20 for B, and has the most items put in the contested pile for those algorithms. But covered here with rank sums $1 + 2 + 7 + 8 = 3 + 4 + 5 + 6 = 18$.

The weighted average is 14.76 (compare 12.00 for the first player in alternating selection, 14.18 for the second).

Works for 35,494 of the 40,320 preferences, 88.0%.

Cost of Equality

88% of S_8 is still not great. May improve for higher n: for 10 items, need rank sums 21 to 30, and 30 has 18 valid 5-part partitions.

Bigger problem is that, even with improving the second player's worst possible outcome, (two data points suggest that) the average equal rank is higher than for alternating selection for even the second player (much higher for first player) .

What about near equality?

Near Equality

Allocate items so that both players' rank sums are within one of each other.

Example: $\pi = (6, 7, 8, 4, 1, 2, 3, 5)$ has $\pi(\{1, 2, 4, 8\}) = \{3, 4, 6, 8\}$ leaving $\{1, 2, 5, 7\}$ for A, so equal rank sum 15 for both. (No lower equal rank sums works.)

But giving B her $\pi(\{1, 2, 3, 4\}) = \{4, 6, 7, 8\}$ leaves $\{1, 2, 3, 5\}$ for A, rank sums 10 and 11, respectively; mean 10.5 compared to 15.

(NN gives $\{1, 2, 3, 4\}$ to A and $\{5, 6, 7, 8\} = \pi(\{1, 2, 3, 8\})$ to B, rank sums 10 & 14. BT gives $\{1, 2, 3\}$ to A and $\{6, 7, 8\}$ to B with contested pile $\{4, 5\}$. SS and AL give same as near-equal outcome.)

Near Equality

For $2n = 6$, this covers $(1, 2, 3, 4, 5, 6)$ with, e.g., $1 + 3 + 6 = 10$ and $2 + 4 + 5 = 11$.

Near equality gives divisions for 682 preferences of 720, 94.7%, with an average rank sum of 7.26 (compare alternating selection averages 7.00 for A and 8.60 for B).

For $2n = 8$, near equality gives divisions for 39,997 preferences of 40,320, 99.2%, with an average rank sum of 13.63 (compare alternating averages 12.00 for A and 14.18 for B).

The remaining 0.8% include $(1, 2, 3, 8, 7, 6, 5, 4)$ and $(5, 3, 4, 2, 1, 8, 6, 7)$ and $(8, 6, 4, 5, 3, 2, 1, 7)$ and 320 more.

Suggestion and Many Questions

Near equal rank sum (within one) or "fuzzy equality" divisions might be a good compromise from requiring strict equality: Data suggest that it gives divisions for almost all preferences and the average near equal rank sums fall between the averages for the first and second player in alternating selection.

- How often do NN, SS, BT, AL give equal or near equal rank sum divisions?
- Better programming (and program?) to get more data.
- Characterization of preferences that do not give equal or near equal rank sum divisions.
- Strife measure on preferences that lead to problems (large contested piles, no equal rank sum division, etc.).