Definition

Let $\mathfrak{B}$ be a Boolean algebra. A function $\mu : \mathfrak{B} \to [0, \infty)$ is called a **measure** if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathfrak{B}$ and $A \cap B = \emptyset$.

Theorem (Stone Representation Theorem)

If $\mathfrak{B}$ is a Boolean algebra, then there exists a Stone Space Ω such that $\mathfrak{B}$ is Boolean isomorphic to $cl(\Omega)$. Furthermore, if Ω_1 and Ω_2 are two such Stone Spaces, then Ω_1 is homeomorphic to Ω_2.

Because Stone Spaces representing a Boolean algebra $\mathfrak{B}$ are unique up to a homeomorphism, we call any such Stone space the Stone space of $\mathfrak{B}$.

Definition

If $\mathfrak{B}$ is a σ–algebra of subsets of a set Ω and μ is a measure on $\mathfrak{B}$ μ is called a continuous measure if whenever $\{A_n\}$ is a decreasing sequence in $\mathfrak{B}$ such that

- if $\cap_{n=1}^{\infty} A_n = \emptyset$
- then $\lim_{n\to\infty} \mu(A_n) = 0$.

Definition

A Boolean algebra $\mathfrak{B}$ is called a measure algebra if there exists a finitely additive measure μ on $\mathfrak{B}$ such that μ is strictly positive and continuous.

Theorem (Gaifman, 1956)

There exists a Boolean algebra without any strictly positive finitely additive measure.

Definition

Suppose $\mathfrak{B}$ is a Boolean algebra If $\mathfrak{C} \subset \mathfrak{B}$ then $\mathfrak{C}$ is called a (pairwise) disjoint collection if $a, b \in \mathfrak{B}$ implies $a \wedge b = 0$. A disjoint collection $\mathfrak{C}$ is called a partition if $\bigvee \mathfrak{C} = 1$. If every disjoint collection in $\mathfrak{B}$ is countable, $\mathfrak{B}$ is said to have the countable chain condition (c.c.c.)

Definition

Suppose $\mathfrak{B}$ is a σ–algebra with c.c.c. If π is a partition in $\mathfrak{B}$ and $a \in \mathfrak{B}$, we say that a is finitely covered by π if there is a finite set $\{a_1, \ldots, a_n\} \subset \pi$ such that

$$a \leq \bigvee \{a_n : 1 \leq i \leq n\}.$$

We say that $\mathfrak{B}$ satisfies the generalized distributive law (g.d.l) if whenever $\{\pi_n\}$ is a sequence of partitions, there is a single partition π such that if $a \in \pi$, a is finitely covered by each π_n.

In 1937 John von Neumann posed a famous problem in the Scottish Book (Problem 163).

Problem (von Neumann, 1937)

If $\mathfrak{B}$ is a Boolean algebra that satisfies the c.c.c. and g.d.l., is $\mathfrak{B}$ a measure algebra ?

Theorem

If $\mathfrak{B}$ is a Boolean σ–algebra with the g.d.l. and c.c.c. and $\mathfrak{B}$ admits a strictly positive finitely additive measure then $\mathfrak{B}$ is a measure algebra.

Proof.

Note that c.c.c. follows automatically from the existence of a strictly positive finitely additive measure. Let φ denote this finitely additive measure. For each $a \in \mathfrak{B}$ define

$$\varphi^\star(a) = \inf_\pi \sum_{a \in \pi} \varphi(a)$$

where the infimum is taken over all partitions π of a (π is a partition of a if $\pi \cup \{a'\}$ is a partition). It is easily verified that $\varphi^\star$ is strictly positive. To see that let $a \in \mathfrak{B} \setminus \{0\}$ and suppose that partitions π_a of a are chosen so that

$$\varphi^*(a) = \lim_{n \to \infty} \sum_{b \in \pi_n} \varphi(b).$$

Since $\mathfrak{B}$ satisfies the g.d.l. there exists $c \in \mathfrak{B}$ such that $c \neq 0, c \leq a$ and c is finitely covered by each π_n. Then

$$\varphi(c) \leq \sum_{b \in \pi_n} \varphi(b)$$

and thus $0 < \varphi(c) \leq \varphi^*(a)$. It now follows that $\mathfrak{B}$ is a measure algebra.

Problem

If $\mathfrak{B}$ is a Boolean algebra with the c.c.c., does $\mathfrak{B}$ admit a strictly positive finitely additive measure ?

Note : The example of Gaifman is of a Boolean algebra **without** the c.c.c. *i.e.* it is an example of a Boolean algebra without any strictly positive finitely additive measure but not satisfying the conditions of the Problem.

Definition

Suppose that $\mathfrak{B}$ is a Boolean algebra and $\nu : \mathfrak{B} \to [0, \infty)$ such that

- (1) $\nu(0) = 0$,
- (2) if $A, B \in \mathfrak{B}$ and $A \leq B$ then $\nu(A) \leq \nu(B)$, (ν is monotone)
- (3) if $A, B \in \mathfrak{B}$ then $\mu(A \vee B) \leq \mu(A) + \mu(B)$, ($\mu$ is subadditive)

then ν is called a submeasure on $\mathfrak{B}$.

Definition

If $\mathfrak{B}$ is a σ–algebra of subsets of a set Ω and ν is a submeasure on $\mathfrak{B}$ then ν is calles a continuous submeasure if $\lim_{n\to\infty} \nu(A_n) = 0$ whenever $\{A_n\}$ is a sequence in $\mathfrak{B}$ such that $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Definition

Suppose ν_1 and ν_2 are submeasures on a Boolean algebra $\mathfrak{B}$. We say that ν_2 is absolutely continuous with respect to ν_1 if for any sequence $\{A_n\}$ in $\mathfrak{B}$ $\lim_{n\to\infty} \nu_1(A_n) = 0$ implies that $\lim_{n\to\infty} \nu_2(A_n) = 0$. ν_1 is equivalent to ν_2 if ν_1 is absolutely continuous with respect to ν_2 and ν_2 is absolutely continuous with respect to ν_1.

MAHARAM PROBLEM I

Suppose $\mathfrak{B}$ is a σ–algebra of subsets of a set Ω. If ν is a continuous submeasure on $\mathfrak{B}$, does there exist a finite measure on $\mathfrak{B}$ equivalent to ν ?

Definition

A Boolean algebra $\mathfrak{B}$ is called a submeasure algebra if there exists a submeasure ν on $\mathfrak{B}$ such that

- (1) For every $a \in \mathfrak{B} \setminus \{0\}$, $\nu(a) > 0$ (ν is strictly positive),
- (2) If $\{a_n\}$ is a sequence in $\mathfrak{B}$ such that $a_1 \geq a_2 \geq \dots$ and $\wedge\{a_n : n \in \mathbb{N}\} = 0$ then $\lim_{n\to\infty} \nu(a_n) = 0$ (ν is a continuous submeasure).

MAHARAM PROBLEM II

Is every submeasure algebra a measure algebra ?

Example (D.Maharam, 1947)

There exists a Boolean algebra $\mathfrak{B}$ with the c.c.c. and the g.d.l. such that $\mathfrak{B}$ is not even a submeasre algebra. The algebra was constructed using the negation of the Suslin Hypothesis.

Definition

Suppose that $\mathfrak{B}$ is a Boolean algebra and $\mathfrak{F} \subset \mathfrak{B}$. $\mathfrak{F}$ has an intersection number λ $((\mathfrak{F}) = \lambda)$ for $\lambda \geq 0$ if λ is the largest nonnegative integer so that whenever $\{a_1, \ldots, a_n\}$ is a finite sequence in $\mathfrak{F}$, then there exists a finite subsequence $i_1 < \cdots < i_K$ with $k \geq \lambda$ such that $a_1 \wedge a_2 \wedge \cdots \wedge a_{i_K} \neq 0$.

Theorem

Suppose $\mathfrak{B}$ is a Boolean algebra and $\mathfrak{F} \subset \mathfrak{B}$. For $\lambda \geq 0$, $i(\mathfrak{F}) = \lambda$ if and only if there is a finitely additive probability measure φ on $\mathfrak{B}$ such that $\varphi(a) \geq \lambda$ for all $a \in \mathfrak{F}$.

Theorem (J.F.Kelley)

A Boolean algebra $\mathfrak{B}$ is a measure algebra if and only if $\mathfrak{B}$ is a $\sigma-$algebra such that

- *(i) $\mathfrak{B}$ satisfies the g.d.l.*
- *(ii) $\mathfrak{B} \setminus \{0\} = \bigcup_{n=1}^{\infty} \mathfrak{F}_n$, where $i(\mathfrak{F}_n) > 0$ for each n.*

NOTE

(ii) characterize Boolean algebras which possess a strictly positive finitely additive probability measure.

Suppose every submeasure algebra is a measure algebra. Let ν be a strictly positive continuous submeasure on a Boolean σ–algebra $\mathfrak{B}$ and let μ be an equivalent probability measure. For every $\epsilon > 0$ there is a $\delta > 0$ so that, if $\mu(a) < \delta$, then $\nu(a) < \epsilon$. If we let

$$\mathfrak{F} = \{a \in \mathfrak{B} : \nu(a) \geq \epsilon\},$$

$a \in \mathfrak{F}$ implies $\mu(a) \geq \delta$. Thus $i(\mathfrak{F}) \geq \delta$. Thus we obtain another formulation of the Maharam's Problem.

MAHARAM PROBLEM III

If ν is a continuous strictly positive submeasure on a Boolean algebra $\mathfrak{B}$ and $\epsilon > 0$ is

$$i(\{a \in \mathfrak{B} : \nu(a) \geq \epsilon\} > 0?$$

Definition

A submeasure ν on a Boolean algebra $\mathfrak{B}$ is pathological if for some $\epsilon > 0$

$$i(\{a \in \mathfrak{B} : \nu(a) \geq \epsilon\}) = 0.$$

Note that if (without any loss of generality) $\mathfrak{B}$ is an algebra of sets and ν is pathological on $\mathfrak{B}$, then there is a fixed $\epsilon > 0$ so that if $\lim_{n\to\infty} \delta_n = 0$ with $\delta_n > 0$ there exists a finite sequence $\{A_{in}, \ldots, A_{K_n n}\}$ so that

$$\sum_{i=1}^{K_n} 1_{A_{in}} \leq \delta_n K_n$$

but $\nu(A_{in}) \geq \epsilon$. So if ν is pathological on $\mathfrak{B}$ it is pathological on the countable algebra generated by the sets $\{A_{iK}\}$,*i.e.* ν is pathological on a countable subalgebra.

MAHARAM PROBLEM IV

If ν is a continuous strictly positive submeasure on a Boolean algebra $\mathfrak{B}$ and $\epsilon > 0$ is

$$i(\{a \in \mathfrak{B} : \nu(a) \geq \epsilon\}) > 0?$$

Corollary (Christensen–Herer)

A submeasure ν is pathological if and only if

$$\nu(a) \geq \mu(a),\ a \in \mathfrak{B}$$

for a measure $\mu : \mathfrak{B} \to [0, \infty)$ implies that μ is **not** *strictly positive,i.e. a pathological submeasure does not majorize any strictly positive measure.*

Examples of pathological submeasures were constructed by Herer and Christensen, Popov, Talagrand *et al.* They all are non–exhaustive, *i.e.* they are not counterexamples to the **Maharam Problem**.

Control Measure Problem

Let X be a vector space over the real numbers and suppose that $\|\cdot\|$ is a nonnegative real valued function on X. Suppose also that

- (1) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,
- (2) $\|\alpha x\| \leq \|x\|$ when $x \in X$ and α is a real number with $|\alpha| \leq 1$,
- (3) $\|x\| = 0$ implies $x = 0$,
- (4) if $x \in X$ and $\{\alpha_n\}$ is a sequence of scalars such that $\lim_{n\to\infty} \alpha_n = 0$ then $\lim_{n\to\infty} \|\alpha_n x\| = 0$.

Then $\|\cdot\|$ is called an F–norm on X. The metric topology defined by $d(x,y) = \|x-y\|$ provides X with linear topology.

Suppose $\mathfrak{B}$ is an algebra of subsets of a set Ω and $\mu : \mathfrak{B} \to X$ such that for every pair of disjoint sets $A, B \in \mathfrak{B}$

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Then μ is a finitely additive X–valued vector measure. The measure μ is said to be exhaustive if, whenever $\{A_n\}$ is a pairwise disjoint collection in $\mathfrak{B}$

$$\lim_{n\to\infty} \mu(A_n) = 0.$$

A finitely additive measure φ on $\mathfrak{B}$ is called a control measure for μ if $\lim_{n\to\infty} \varphi(A_n) = 0$ implies $\lim_{n\to\infty} \|\mu(A_n)\| = 0$ for every sequence $\{A_n\}$ in $\mathfrak{B}$.

Define a submeasure ν on $\mathfrak{B}$ by

$$\nu(A) = \sup\{\|\mu(B)\| : B \subset A,\ B \in \mathfrak{B}\}.$$

Note that ν is exhaustive if (and only if) μ is exhaustive.

Theorem (Bartle–Dunford–Schwartz)

Let X be a Banach space and $\varphi : \mathfrak{B} \to X$ a countably additive vector measure. There exists a control measure $\mu : \mathfrak{B} \to [0, \infty)$ for φ.

Theorem (Rybakov)

Let X be a Banach space and let $\varphi : \mathfrak{B} \to X$ be a countably additive vector measure. There exists a linear continuous functional x' on X such that

$$|x' \circ \varphi| : \mathfrak{B} \to [0, \infty), \textit{defined as} \, |x'(\varphi(a))|,$$

is a control measure for φ.

MAHARAM PROBLEM V (Control Measure Problem)
Does every F-space valued countably additive measure have a control measure ?

Definition

If ν is a submeasure on a Boolean algebra $\mathfrak{B}$, we say that ν is uniformly exhaustive if for every $\epsilon > 0$ there exists a positive integer N such that whenever $\{a_1, \ldots, a_N\}$ is a pairwise disjoint collection in $\mathfrak{B}$, there exists some a_i with $\nu(a_i) < \epsilon$.

Of course, uniformly exhaustive submeasures are exhaustive. It is easy to see that ν is uniformly exhaustive if and only if there exists a decreasing sequence $\{\epsilon_n\}$ of positive numbers so that $\lim_{n\to\infty} \epsilon_n = 0$ and whenever $\{a_n\}$ is a pairwise disjoint sequence in $\mathfrak{B}$ with the sequence $\{\nu(a_n)\}$ decreasing, then $\nu(a_n) \leq \epsilon_n$ for each n. For instance, if ν is a finitely additive probability measure we can take $\epsilon_n = \frac{1}{n}$. It is also easy to see that if a submeasure ν has an equivalent finitely additive measure, then ν is uniformly exhaustive.

Theorem (Talagrand)

A submeasure is uniformly exhaustive if and only if it has an equivalent finitely additive measure.

MAHARAM PROBLEM VI

Is every exhaustive submeasure uniformly exhaustive ?

Theorem (Talagrand)

There exists a non–zero exhaustive submeasure ν on the algebra $\mathfrak{B}$ of clopen subsets of the Cantor set that is not uniformly exhaustive (and thus not absolutely continuous with respect to a measure). Moreover, no non–zero measure μ on $\mathfrak{B}$ is absolutely continuous with respect to ν.

Corollary (1)

There exists a submeasure algebra $\mathfrak{B}$ that is not a measure algebra. In fact, not only there is no strictly positive measure on $\mathfrak{B}$, but there is no non–zero continuous measure on it.

Corollary (2)

There exists a σ–complete algebra that satisfies the c.c.c. and the g.d.l. but is not a measure algebra.

Corollary (3)

There exists an exhaustive vector–valued measure that does not have a control measure.

Let ν be a counterexample to the Maharam Problem. Then ν is a continuous submeasure on a σ–algebra of sets $\mathfrak{B}$ of a set Ω. Then the space of measurable functions (actually equivalence classes of functions) $L_0(\nu)$ can be provided with an F–norm so that

$$\lim_{n\to\infty} f_n = 0 \text{ if and only if}$$

for every $\epsilon > 0$

$$\lim_{n\to\infty} \nu\{x : |f_n(x)| \geq \epsilon\} = 0.$$

The vector measure $\mu : \mathfrak{B} \to L_0(\nu)$ defined by

$$\mu(A) = 1_A$$

is exhaustive. The measure μ has no control measure.