Tiling with fences

16:00 Wednesday 26th October 2016, Room 3303, MUIC

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We look at tiling an n-board (a linear array of n square cells of unit width) using square tiles and $(\frac{1}{2}, g)$-fence tiles where $g \in \mathbb{Z}^+$. A $(\frac{1}{2}, g)$-fence is composed of two pieces of width $\frac{1}{2}$ separated by a gap of width g. Although originally introduced to give a new combinatorial interpretation of the Tribonacci numbers, tiling with fences and squares can also be used to describe strongly restricted permutations in a natural and intuitive way.

Acknowledgements

Ken Edwards

- Edwards K (2008/2009)

  "A Pascal-like triangle related to the Tribonacci numbers".

  Fibonacci Quarterly, **46**/**47**(1), 18–25.

- Edwards K, Allen MA (2015)

  "Strongly restricted permutations and tiling with fences".

  Discrete Applied Mathematics, **187**, 82–90.

- Benjamin AT, Quinn JJ (2003)

  Proofs That Really Count: The Art of Combinatorial Proof, MAA.

Outline

- Combinatorial proof
- Tiling an n-board with black and white squares $\longrightarrow$ binomial coefficients
- Tiling an n-board with squares and dominoes $\longrightarrow$ Fibonacci numbers
- Tiling an n-board with squares, dominoes, and triominoes $\longrightarrow$ Tribonacci numbers
- Tiling an n-board with squares and $(\frac{1}{2}, 1)$-fences $\longrightarrow$ Tribonacci numbers
- Strongly restricted permutations
- Tiling an n-board with squares and various $(\frac{1}{2}, g \in \mathbb{Z}^+)$-fences

Combinatorial proof of the Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

PROOF: Write

$$(x+y)^n = \overbrace{(x+y)(x+y)\cdots(x+y)}^{n \text{ times}}$$

and count the number of ways of creating $x^k y^{n-k}$. $\square$

Tiling an n-board with black and white squares

The number of ways to tile an n-board using k black squares (and $n-k$ white squares) is $\binom{n}{k}$.

A well-known identity:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

PROOF: The LHS is the total number of ways of tiling an n-board with black and white squares. Since we can choose whether each cell on the board is black or white independently, the number of ways of doing this is

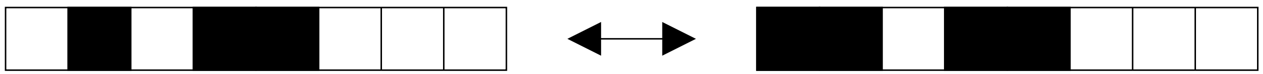
$$\overbrace{2 \times 2 \times \cdots \times 2}^{n \text{ times}}.$$

□

An example of a bijective proof:

$$\sum_{k \geqslant 0} \binom{n}{2k} = \sum_{k \geqslant 0} \binom{n}{2k+1}$$

PROOF: Call a tiling of an n-board an *even tiling* (*odd tiling*) if k, the number of black squares used, is even (odd). Any even (odd) tiling can be mapped to an odd (even) tiling by flipping the colour of the first square.



Thus the number of even tilings equals the number of odd tilings. □

The basic Pascal triangle identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

PROOF: Consider the last cell of an n-board tiled with k black tiles. If the cell is black (is white), the number of ways to tile the remaining $n-1$ cells is $\binom{n-1}{k-1}$ (is $\binom{n-1}{k}$). □

$n \setminus k$	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

Tiling an n-board with squares and dominoes

Let f_n be the number of ways to tile an n-board with squares and dominoes. [This is the same as the number of compositions of n containing only 1's and 2's. Thus, e.g., $f_4 = 5$ since 4=1+1+1+1=1+1+2=1+2+1=2+1+1=2+2.] Also let $f_0 = 1$, $f_{-1} = 0$. Then

$$f_n = f_{n-1} + f_{n-2}$$

PROOF: If the last tile is a square (a domino), there are f_{n-1} (are f_{n-2}) ways to tile the remaining $n-1$ cells ($n-2$ cells). □

Thus $f_n = F_{n+1}$ where F_n is the nth Fibonacci number defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ giving the famous sequence 0, 1, 1, 2, 3, 5, 8, ...

An example of a combinatorial proof of a Fibonacci number identity

$$f_{m+n} = f_m f_n + f_{m-1} f_{n-1}$$

PROOF: Consider a tiled $(m+n)$-board. The mth cell is either occupied by the first half of a domino or is not. If the latter, there are f_m ways to tile the cells from 1 to m and f_n ways to tile the remaining cells (from cell $m+1$ to $m+n$) which makes $f_m f_n$ ways in total for the whole board. If the mth cell is the first half of a domino, there are f_{m-1} to tile the cells before that domino and f_{n-1} ways to tile the cells after that domino, making $f_{m-1} f_{n-1}$ ways in total for the whole board. $\square$

Tiling an n-board with squares, dominoes, and triominoes

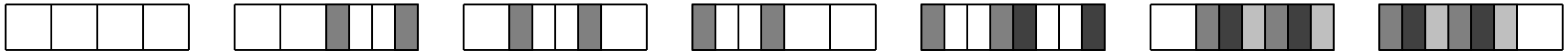
Let t_n be the number of ways to tile an n-board using squares, dominoes, and triominoes. Take $t_0 = 1$, $t_{n<0} = 0$. Then

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}$$

PROOF: If the last tile has length m, the number of ways to tile the remaining $n - m$ cells is t_{n-m}. □

Thus $t_n = T_{n+1}$ where T_n is the nth Tribonacci number defined by $T_n = T_{n-1} + T_{n-2} + T_{n-3}, T_1 = 1, T_{n \leqslant 0} = 0$ which gives the sequence 0, 1, 1, 2, 4, 7, 13, 24, ...

Tiling an n-board with squares and $(\frac{1}{2}, 1)$-fences



Any tiling can be expressed as a tiling with metatiles. A *metatile* is is a minimal arrangement of tiles that exactly covers an integral number of adjacent cells. [It is minimal in the sense that if one or more tiles exactly covering a set of adjacent cells are removed from a metatile then the result is no longer a metatile.]

With squares and $(\frac{1}{2}, 1)$-fences there are 3 types of metatiles: (i) free square;
(ii) filled fence (a fence filled by a non-aligned square);
(iii) trifence (3 interlocking fences).

Let t_n be the number of ways to tile an n-board using squares and $(\frac{1}{2}, 1)$-fences. Take $t_0 = 1, t_{n<0} = 0$. Then $t_n = t_{n-1} + t_{n-2} + t_{n-3}$.

PROOF: If the last metatile has length m, the number of ways to tile the remaining $n - m$ cells is t_{n-m}. □

A Pascal-like triangle giving $\left[{n \atop k}\right]$, the number of n-board tilings using k $(\frac{1}{2}, 1)$-fences (and $n-k$ squares):

$\left[{n \atop k}\right] = \left[{n-1 \atop k}\right] + \left[{n-2 \atop k-1}\right] + \left[{n-3 \atop k-3}\right]$

PROOF: If the last metatile is of length L and contains K fences, then there are $\left[{n-L \atop k-K}\right]$ ways to tile the remaining $n-L$ cells in a way that uses $k-K$ fences (and $n-L-(k-K)$ squares). □

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	0								
2	1	1	0							
3	1	2	0	1						
4	1	3	1	2	0					
5	1	4	3	3	2	0				
6	1	5	6	5	6	0	1			
7	1	6	10	9	12	3	3	0		
8	1	7	15	16	21	12	6	3	0	
9	1	8	21	27	35	30	14	12	0	1

Strongly restricted permutations

A *strongly restricted permutation* π of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$ is a permutation for which the number of permissible values of $\pi(i) - i$ for each $i \in \mathbb{N}_n$ is less than a finite number K which is independent of n (Lehmer 1970).

Let A_n be the number of restricted permutations. For

$$\pi(i) - i \in W,$$

where W is any finite set of integers which is independent of i and n, a linear recurrence for A_n can be derived via a 5-step procedure (Baltić 2010). E.g., for $W = \{-1, 0, 1, \ldots, r\}$ he got

$$A_n = A_{n-1} + \ldots + A_{n-(r+1)}, \qquad A_0 = 1, \quad A_{n<0} = 0,$$

which is the $(r+1)$-step Fibonacci sequence.

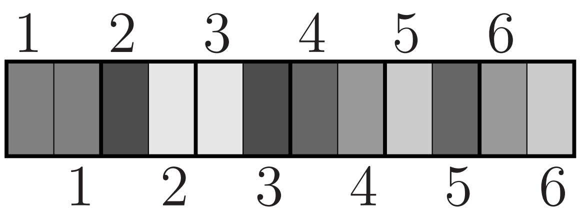
Sloane's Online Encyclopedia of Integer Sequences

`http://oeis.org`

- A000040: 2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,...
- A000105: 1,1,2,5,12,35,108,369,1285,4655,17073,63600,238591,...
- A000602: 1,1,1,2,3,5,9,18,35,75,159,...
- A001203: 3,7,15,1,292,1,1,1,2,1,3,1,14,2,1,1,2,2,2,2,1,84,...
- A002411: 1,6,18,40,75,126,196,288,405,550,...
- A157897: 1,1,0,1,1,0,1,2,0,1,1,3,1,2,0,1,4,3,3,2,0,...
- A080004: 1,1,2,4,9,15,25,46,84,156,280,501,909,1647,2990,...

Bijection between tilings using squares and $(\frac{1}{2}, g \in \mathbb{Z}^+)$-fences and strongly restricted permutations

- An n-board is tiled by n such tiles.
- In all cases, one half of each tile lies on the left side of a cell and the other on the right side of a cell.
- The cell number of the half of a tile which is in the left side of a cell gives the domain i and the codomain $\pi(i)$ is the cell number where the other half of the tile is.



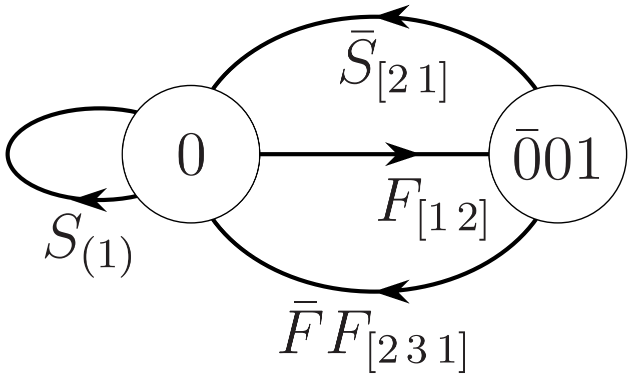
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 5 & 6 & 4 \end{pmatrix}$$

Thus tiling an n-board with squares and $(\frac{1}{2}, 1)$-fences corresponds to the restricted permutation of $1, \ldots, n$ where $\pi(i) - i \in \{-2, -1, 0, 1\}$.

Generating all the possible metatiles using a digraph

- each node represents the next gap (0 = empty half cell; 1 = filled half cell; $\bar{0}$ = gap starts on RHS of cell)
- each arc represents the addition of a tile or tiles at the gap
- each walk starting and ending at the 0 node (and not visiting it in between) corresponds to a metatile

S = aligned square; $\bar{S}$ = non-aligned square;
F = $(\frac{1}{2}, 1)$-fence starting on left side;
$\bar{F}$ = $(\frac{1}{2}, 1)$-fence starting on right side



$[\ldots]$ = part of permutation cycle; $(\ldots)$ = permutation cycle
metatiles: $S_{(1)}$ (= free square); $F\bar{S}_{(12)}$ (= filled fence); $F\bar{F}F_{(123)}$ (= trifence)

Theorem

If A_n is the number permutations of $i = 1, \ldots, n$ satisfying $\pi(i) - i \in \{-1, g_1, \ldots, g_r\}$ where $0 \leqslant g_1 < g_2 < \ldots < g_r$ and $g_r > 0$ then

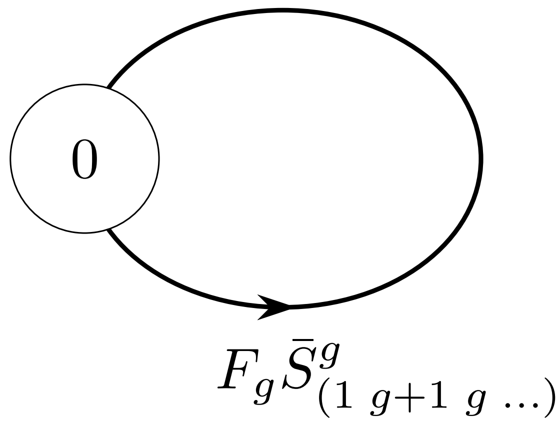
$$A_n = A_{n-(g_1+1)} + \cdots + A_{n-(g_r+1)}, \qquad A_0 = 1, \quad A_{n<0} = 0,$$

and the possible permutation cycles are of the form $(1 \;\; g+1 \;\; g \;\; \ldots \;\; 2)$ for $g = g_l > 0$, $(l = 1, \ldots, r)$.

PROOF: Equivalent to the number of ways to tile an n-board using $\bar{S}$ and $(\frac{1}{2}, g_l)$-fences starting on left side of cell (denoted F_{g_l} and $F_0 \equiv S$).

Lengths of possible metatiles are $g_l + 1$.

Condition on the length of the last metatile to obtain the recursion relation.



□

Tiling with metatiles drawn from an infinite set: definitions

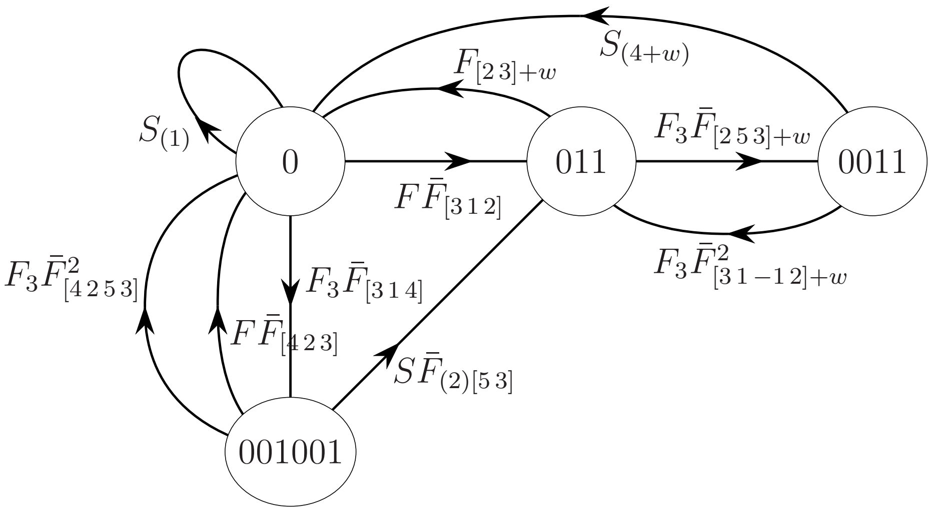
inner cycle does not include the 0-node [011 to 0011 to 011]

common node common to all inner cycles [e.g. 011]

common circuit includes 0-node and common node once each [e.g. 0 to 011 to 0011 to 0]

outer cycle includes 0-node but not the common node [e.g. 0 to 0]

$$\pi(i) - i = \{-2, 0, 1, 3\}$$



Each outer cycle corresponds to a distinct metatile. From each common circuit an infinite family of metatiles is generated via the walk from the 0-node to the common node followed by arbitrary numbers of walks around the inner cycle(s) in any order followed by the walk from the common node to the 0-node.

Theorem: recurrence relation when there is 1 inner cycle

Let $l_{\mathrm{o}i}$ denote the length of the ith outer cycle metatile ($i = 1, \ldots, N_{\mathrm{o}}$) and $l_{\mathrm{c}i}$ denote the length of the metatile corresponding to the ith common circuit ($i = 1, \ldots, N_{\mathrm{c}}$).

If a digraph contains just one inner cycle which contains L tiles then A_n, the number of tilings of an n-board, satisfies

$$A_n = \begin{cases} \displaystyle\sum_{i=1}^{N_{\mathrm{o}}} A_{L-l_{\mathrm{o}i}} + \sum_{i=1}^{N_{\mathrm{c}}} A_{L-l_{\mathrm{c}i}}, & n = L, \\ 1, & n = 0, \\ 0, & n < 0, \\ \displaystyle A_{n-L} + \sum_{i=1}^{N_{\mathrm{o}}} (A_{n-l_{\mathrm{o}i}} - A_{n-l_{\mathrm{o}i}-L}) + \sum_{i=1}^{N_{\mathrm{c}}} A_{n-l_{\mathrm{c}i}}, & \text{otherwise.} \end{cases}$$

Proof

$N_{\rm o}$ = # outer cycles; $l_{{\rm o}i}$ = length of ith outer cycle;

$N_{\rm c}$ = # common circuits; $l_{{\rm c}i}$ = length of ith common circuit;

L = length of inner cycle.

Conditioning on the last metatile,

$$A_n = \sum_{i=1}^{N_{\rm o}} A_{n-l_{{\rm o}i}} + \sum_{i=1}^{N_{\rm c}} \sum_{j=0}^{\infty} A_{n-l_{{\rm c}i}-jL}, \quad n \neq 0, \tag{1}$$

with $A_0 = 1$, $A_{n<0} = 0$. Replacing n by $n - L$ in (1) and subtracting from (1) gives

$$A_n - A_{n-L} = \sum_{i=1}^{N_{\rm o}} (A_{n-l_{{\rm o}i}} - A_{n-l_{{\rm o}i}-L}) + \sum_{i=1}^{N_{\rm c}} A_{n-l_{{\rm c}i}}, \quad n \neq 0,\, n \neq L.$$

A_L is found directly from (1).

Example of a digraph with one inner cycle: Tiling with fixed-point squares, $(\frac{1}{2},1)$-fences (F and $\bar{F}$), and F_3

$L=5$;

common node: 011;

$N_{\rm o}=3$:

S, $F_3\bar{F}\{F\bar{F},F_3\bar{F}^2\}$

where $X\{Y,Z\}$

means XY, XZ

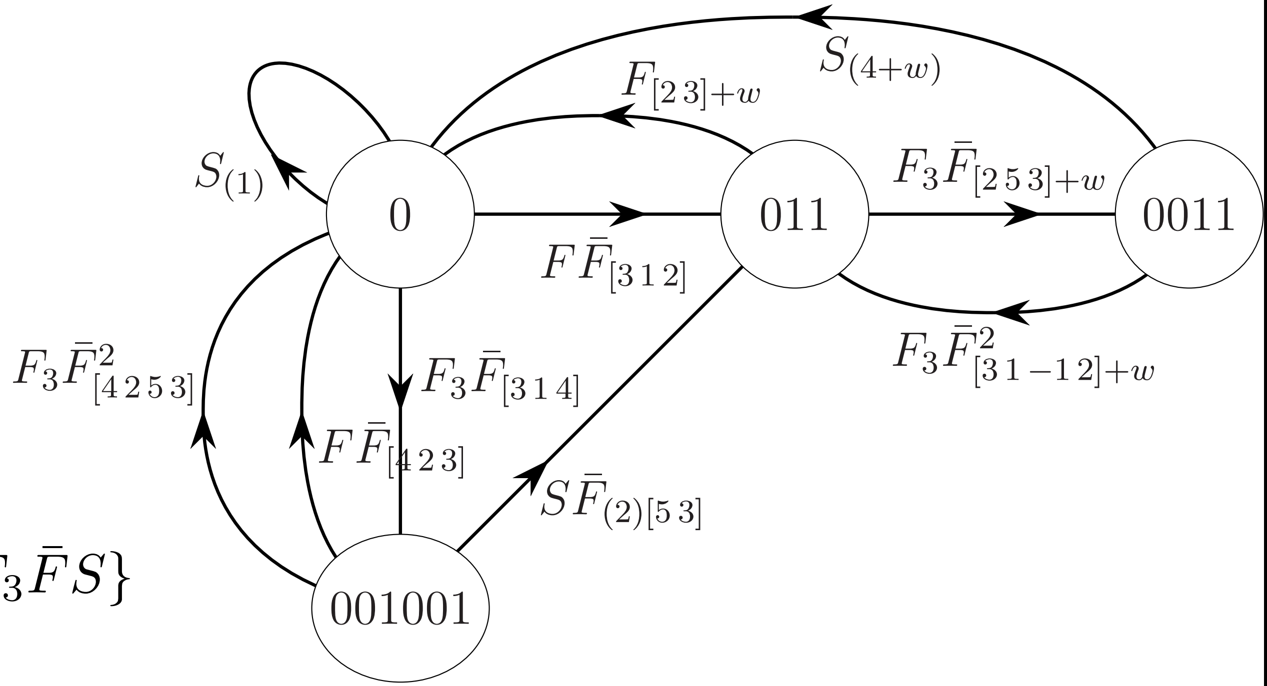
so $l_{{\rm o}i}=1,4,5$;

$N_{\rm c}=4$:

$\{F\bar{F},F_3\bar{F}S\bar{F}\}\{F,F_3\bar{F}S\}$

so $l_{{\rm c}i}=3,5,5,7$.

$\pi(i)-i=\{-2,0,1,3\}$



$A_n=A_{n-1}+A_{n-3}+A_{n-4}+4A_{n-5}-A_{n-6}+A_{n-7}-A_{n-9}-A_{n-10}$,

$A_5=9$ gives $A_{n\geqslant 0}$ = 1,1,1,2,4,9,15,25,46,84,156,... (sequence A080004 in OEIS)

Theorem: recurrence relation when there are N inner cycles and a common node

Let L_r be the number of tiles in the rth inner cycle ($r = 1, \ldots, N$) of a digraph possessing a common node and let $M = \sum_{s=1}^{N} j_s L_s$. Then

$$A_n = \begin{cases} \displaystyle\sum_{i=1}^{N_\mathrm{o}} A_{L_r - l_{\mathrm{o}i}} + \sum_{i=1}^{N_\mathrm{c}} \sum_{j_1,\ldots,j_N \geqslant 0} \binom{j_1 + \cdots + j_N}{j_1, \ldots, j_N} A_{L_r - l_{\mathrm{c}i} - M}, & n = L_r, \\ 1, & n = 0, \\ 0, & n < 0, \\ \displaystyle\sum_{r=1}^{N} A_{n-L_r} + \sum_{i=1}^{N_\mathrm{o}} \left(A_{n-l_{\mathrm{o}i}} - \sum_{r=1}^{N} A_{n-l_{\mathrm{o}i}-L_r} \right) & \\ \qquad \displaystyle + \sum_{i=1}^{N_\mathrm{c}} A_{n-l_{\mathrm{c}i}}, & \text{otherwise} \end{cases}$$

Sketch of proof

Conditioning on the last metatile, we have for $n \neq 0$

$$A_n = \sum_{i=1}^{N_\mathrm{o}} A_{n-l_{\mathrm{o}i}} + \sum_{i=1}^{N_\mathrm{c}} \sum_{j_1,\ldots,j_N \geqslant 0} \binom{j_1 + \cdots + j_N}{j_1, \ldots, j_N} A_{n-l_{\mathrm{c}i}-M} \qquad (2)$$

and $A_0 = 1$, $A_{n<0} = 0$. The multinomial coefficient (which counts the number of arrangements of the inner cycles) results from the fact that changing the order in which the inner cycles are traversed gives rise to distinct metatiles of the same length.

Subtracting $\sum_{r=1}^{N} A_{n-L_r}$ from (2) and using the result for multinomial coefficients that

$$\binom{j_1 + \cdots + j_N}{j_1, \ldots, j_N} = \binom{j_1 + \cdots + j_N - 1}{j_1 - 1, \ldots, j_N} + \cdots + \binom{j_1 + \cdots + j_N - 1}{j_1, \ldots, j_N - 1},$$

gives the required result.

Example of a digraph with 2 inner cycles: Tiling with S, $\bar{F}$, F_2, and F_3

$L_1 = 5$, $L_2 = 2$;

common node: 011;

$N_\mathrm{o} = 4$:

S, $F_2\bar{F}\{S, F_2\bar{F}\}$,

$F_3\bar{F}F_3\bar{F}^2$

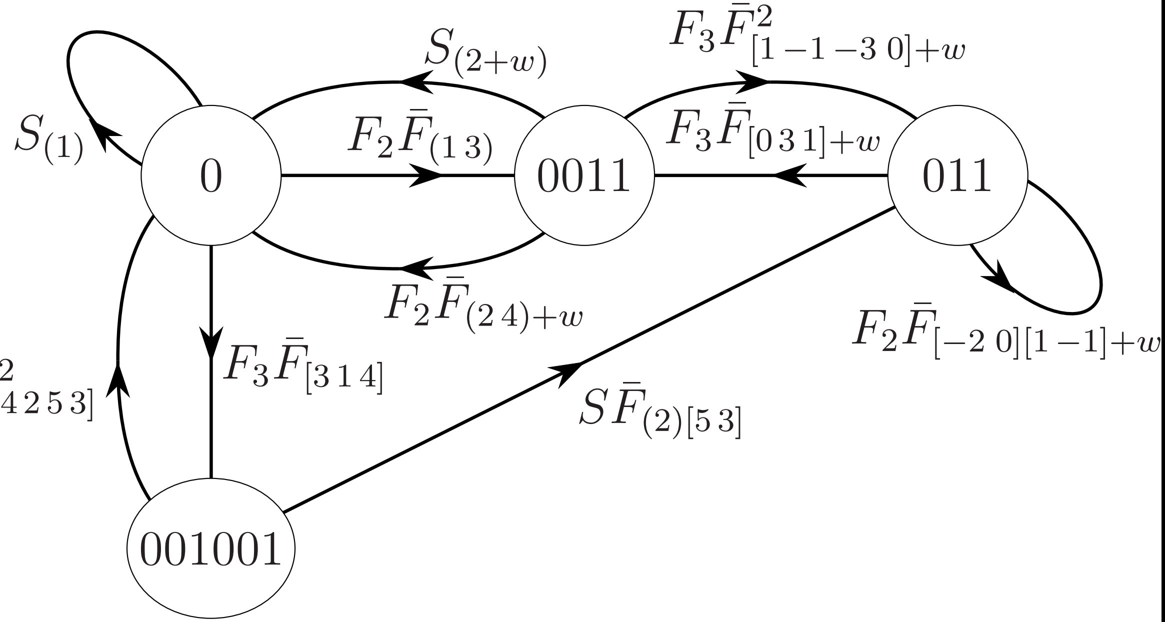
so $l_{\mathrm{o}i} = 1, 3, 4, 5$;

$N_\mathrm{c} = 4$: $\{F_2\bar{F}F_3\bar{F}^2$,

$F_3\bar{F}S\bar{F}\}F_3\bar{F}\{S, F_2\bar{F}\}$

so $l_{\mathrm{c}i} = 3, 5, 5, 7$.

$\pi(i) - i = \{-2, 0, 2, 3\}$



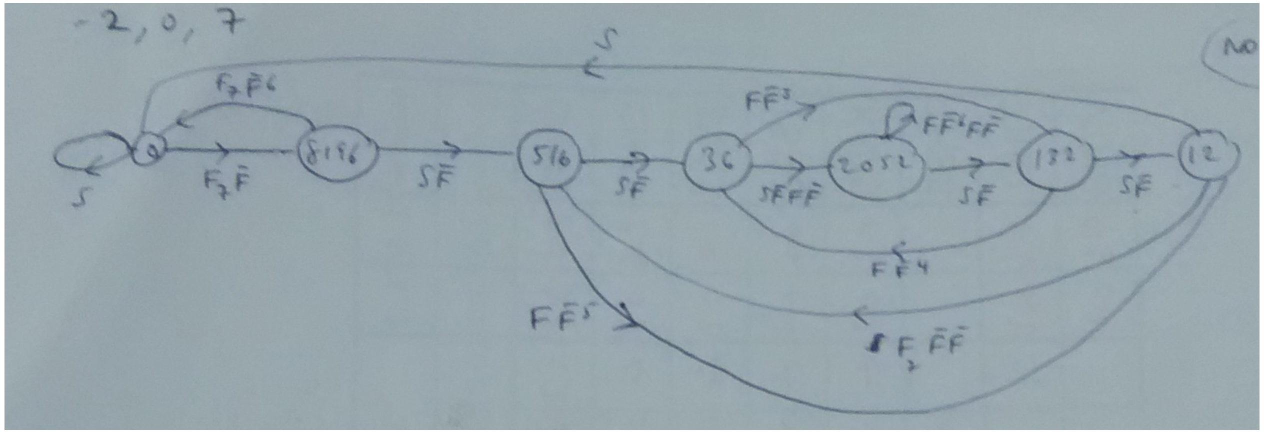
$A_n = A_{n-1} + A_{n-2} + A_{n-4} + A_{n-5} - 2A_{n-6} + A_{n-8} - A_{n-10}$,

$A_2 = 1$, $A_5 = 7$ gives $A_{n\geqslant 0}$ = 1,1,1,2,4,7,11,19,35,62,107,186,328,...

(sequence A080005 in OEIS)

Can we find a recursion relation for all possible tilings?

Not all metatile-generating digraphs with inner cycles have a common node. E.g., $\pi(i) - i \in \{-2, 0, 7\}$:



Counting patterns within strongly restricted permutations

Let $\left[{n \atop k}\right]_{\mathcal{P}}$ be the number of tilings of an n-board which have k instances of a pattern $\mathcal{P}$. Examples of $\mathcal{P}$: S, a type of fence. Let $p_{\mathrm{o}i}$, $p_{\mathrm{c}i}$, and P be the number of instances of $\mathcal{P}$ in the ith outer cycle metatile, the ith common circuit metatile, and the inner cycle, respectively.

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \begin{cases} \displaystyle\sum_{i=1}^{N_\mathrm{o}} \left[{L-l_{\mathrm{o}i} \atop P-p_{\mathrm{o}i}}\right]_{\mathcal{P}} + \sum_{i=1}^{N_\mathrm{c}} \left[{L-l_{\mathrm{c}i} \atop P-p_{\mathrm{c}i}}\right]_{\mathcal{P}}, & n=L \text{ and } k=P, \\ 1, & n=k=0, \\ 0, & n<0,\ k<0, \text{ or } n<k, \\ \displaystyle\left[{n-L \atop k-P}\right]_{\mathcal{P}} + \sum_{i=1}^{N_\mathrm{c}} \left[{n-l_{\mathrm{c}i} \atop k-p_{\mathrm{c}i}}\right]_{\mathcal{P}} & \\ \quad \displaystyle + \sum_{i=1}^{N_\mathrm{o}} \left(\left[{n-l_{\mathrm{o}i} \atop k-p_{\mathrm{o}i}}\right]_{\mathcal{P}} - \left[{n-l_{\mathrm{o}i}-L \atop k-p_{\mathrm{o}i}-P}\right]_{\mathcal{P}} \right), & \text{otherwise.} \end{cases}$$

Permutation cycles

Do arbitrarily long metatiles correspond to arbitrarily long permutation cycles?

whole-cell node node whose gaps and filled regions are aligned with cell boundaries. The simplest whole-cell nodes are therefore 0 and 0011.

Lemma 1: A permutation cycle corresponding to all or part of a metatile is completed only when the corresponding arc enters a whole-cell node.

Corollary 1: Permutation cycles of arbitrary length are possible for a strongly restricted permutation if and only if the corresponding digraph contains at least one inner cycle which lacks a whole-cell node.

Further references

- V. Baltić (2010) On the number of certain types of strongly restricted permutations, Appl. Anal. Discrete Math. 4 (1) 119–135.
- D. H. Lehmer (1970) Permutations with strongly restricted displacements, in: Combinatorial Theory and its Applications II (Proceedings of the Colloquium, Balatonfured, 1969), North-Holland, Amsterdam, pp. 755–770.

THANK YOU FOR YOUR ATTENTION