CMC surfaces and the Willmore–like functional in the Heisenberg group and some other Thurston geometries

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Theorem (Hopf)

Any immersed CMC (Constant Mean Curvature) sphere in $\mathbb{E}^3$ (the three–dimensional Euclidean space) is a round sphere.

Theorem (Alexandrov)

Any embedded closed CMC surface in $\mathbb{E}^3$ is a round sphere.

Theorem (Wente; Abresch)

There exist immersed CMC tori in $\mathbb{E}^3$.

Willmore Conjecture (Marques,Neves)

Any compact surface Σ of genus one in $\mathbb{E}^3$ must satisfy $W(\Sigma) = \int_\Sigma H^2 ds \geqslant 2\pi^2$ and the minimum is attained exactly for scales of the Clifford torus:

$$T = \{((\sqrt{2} + \cos u)\cos v, (\sqrt{2} + \cos u)\sin v, \sin u) \mid u, v \in \mathbb{R}^2\}$$

Theorem

Any maximal, simply connected, 3–dimensional geometry that admits a compact quotient is equivalent to one of the geometries $(M, \mathrm{Iso}(M))$, where M is one of the Riemannian manifolds: $\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \widetilde{\mathrm{SL}_2}(\mathbb{R}), \mathrm{Nil}$ and Sol.

The model spaces for Nil, $\widetilde{\mathrm{SL}_2}(\mathbb{R})$, $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$

They belong to the family of Riemannian 3–manifolds $E(k, \tau)$, $k \in \mathbb{R}$, $\tau \in \mathbb{R}$ which are as follows. If $k \geqslant 0$ then $E(k, \tau)$ is $\mathbb{R}^3$ with the metric:

$$ds^2 = \frac{dx^2 + dy^2}{\left(1 + \frac{k}{4}(x^2 + y^2)\right)^2} + \left(dz + \frac{\tau(ydx - xdy)}{1 + \frac{k}{4}(x^2 + y^2)}\right)^2.$$

If $k < 0$, then $E(k, \tau)$ is the product $\mathrm{D}^2(\frac{2}{\sqrt{-k}}) \times \mathbb{R}$ with the metric, where $\mathrm{D}^2(\frac{2}{\sqrt{-k}}) = \{(x, y) \,|\, x^2 + y^2 < \frac{4}{-k}\}$. The family $E(k, \tau)$ is also referred to as Bianchi–Cartan–Vranceanu family

Tomter:

There exist embedded rotationally invariant CMC–spheres in the three–dimensional Heisenberg group.

Abresch, Rosenberg:

Any immersed CMC sphere in $E(k, \tau)$ is actually one of the embedded rotationally invariant CMC–spheres.

Daniel:

An analog of the Bonnet theorem for $E(k, \tau)$.

Some References:

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A Weierstrass representation:

$$x^1(z,\bar{z}) = \int_{z_0}^{z} Z_1 dz + \overline{Z}_1 d\bar{z},$$

$$x^2(z,\bar{z}) = \int_{z_0}^{z} Z_2 dz + \overline{Z}_2 d\bar{z},$$

$$x^3(z,\bar{z}) = \int_{z_0}^{z} (Z_3 dz + \overline{Z}_3 d\bar{z}) + \frac{1}{2}\int_{z_0}^{z} x^1(Z_2 dz + \overline{Z}_2 d\bar{z}) -$$

$$\frac{1}{2}\int_{z_0}^{z} x^2(Z_1 dz + \overline{Z}_1 d\bar{z}),$$

$$Z_1 = \frac{i}{2}(\overline{\psi}_2^2 + \psi_1^2), \quad Z_2 = \frac{1}{2}(\overline{\psi}_2^2 - \psi_1^2), \quad Z_3 = \psi_1 \overline{\psi}_2.$$

For a proper choice of a conformal parameter z, any CMC surface satisfies:

$$\partial_z \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} v_z & -e^{-v} \\ -e^{v} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$\partial_{\bar z} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 & e^{v} \\ e^{-v} & v_{\bar z} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

Theorem

Each nonzero constant mean curvature surface, in some neighborhood of a non–umbilical point, corresponds to some solution $v = \rho + i\varphi$ to the following system:

$$v_{z\bar z} + 2\sinh 2v = 0,$$

$$\frac{\varphi_x^2}{(\cosh\rho)^2} + \frac{\varphi_y^2}{(\sinh\rho)^2} = 8\left(\cos 2\varphi - \mathrm{Re}\,\frac{2H+i}{2H-i}\right).$$

Theorem

Each minimal surface, in some neighborhood of a non–umbilical point, corresponds to some solution $v = \rho + i\varphi$ to the following system:

$$v_{z\bar{z}} + 2\sinh 2v = 0,$$

$$\mathrm{Re}[e^v] = 0.$$

After a proper change of variables we obtain the electrolyte equation:

$$\Delta\sigma - \sinh\sigma = 0,$$

where σ is a real–valued function.

Let us make the following change of variables: $v = -\frac{u}{2}$, $\widehat{\psi}_1 = i\psi_1$, $\widehat{\psi}_2 = e^{-v}\psi_2$ and $z = \frac{1}{2}w$. We have:

$$\partial_w \begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix} = \begin{pmatrix} -\frac{u_w}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{u_w}{2} \end{pmatrix} \begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix},$$

$$\partial_{\bar{w}} \begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{i}{2}e^{-u} \\ -\frac{i}{2}e^{u} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix}.$$

The system above allows to introduce a spectral parameter:

$$\partial_w \begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix} = \begin{pmatrix} -\frac{u_w}{2} & -\frac{i}{2} \\ -\frac{i}{2}\lambda & \frac{u_w}{2} \end{pmatrix} \begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix},$$

$$\partial_{\bar{w}} \begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{i}{2\lambda} e^{-u} \\ -\frac{i}{2} e^{u} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix}.$$

Let Γ be a Riemannian surface and λ be a meromorphic function on Γ. Suppose that λ has two–order zero at a point $P_1 \in \Gamma$ and two–order pole at a point $P_2 \in \Gamma$. Let k_1^{-1} and k_2^{-1} be the local parameters in neighborhoods of P_1 and P_2 such that: $\lambda = k_1^{-2}$ in a neighborhood of P_1 and $\lambda = k_2^2$ in a neighborhood of P_2. Suppose that $D = \gamma_1 + \dots + \gamma_g$ is a divisor on Γ, where g is the genus of Γ.

Definition

A Baker–Akhiezer function, corresponding to the spectral data:

$$\{\Gamma, P_1, P_2, k_1^{-1}, k_2^{-1}, D\},$$

is a meromorphic function $\psi(w, \bar{w}) : \Gamma \to \mathbb{C}$ with the following properties:

1) ψ is a meromorphic on $\Gamma \backslash \{P_1, P_2\}$ has simple poles at the divisor $D = \gamma_1 + \dots + \gamma_g$;

2) in neighborhoods of P_1 and P_2 the function ψ has essential singularities such that the functions $\psi \exp(\frac{i}{2} k_1 \bar{w})$ and $\psi \exp(\frac{i}{2} k_2 w)$ are holomorphic in these neighborhoods.

For a given spectral data there exists a Baker–Akhiezer function which is unique, up to multiplication by a constant. Then, there exist unique function $\widehat{\psi}_1(w, \bar{w})$ which has simple poles at $\gamma_1, \ldots, \gamma_g$ and has the asymptotics at the points P_1 and P_2 as follows:

$$\widehat{\psi}_1(w, \bar{w}) = exp\left(-\frac{i}{2}k_1\bar{w}\right)\left(c_1(w, \bar{w}) + \frac{d_1(w, \bar{w})}{k_1} + o(k_1^{-1})\right),$$

$$\widehat{\psi}_1(w, \bar{w}) = exp\left(-\frac{i}{2}k_2 w\right)\left(1 + \frac{f_1(w, \bar{w})}{k_2} + o(k_2^{-1})\right).$$

Let us define $\widehat{\psi}_2(w, \bar{w})$ as:

$$\widehat{\psi}_2(w, \bar{w}) = 2i\left(\psi_{1w} - \psi_1\frac{c_{1w}}{c_1}\right).$$

Proposition

Suppose that $u = \ln\left(-\frac{i}{2f_{1\bar{w}}}\right)$, then the vector function $\begin{pmatrix}\widehat{\psi}_1\\ \widehat{\psi}_2\end{pmatrix}$ is a solution of the system for $\widehat{\psi}_1$ and $\widehat{\psi}_2$ above.

Proposition

Suppose that Γ admits an anti–holomorphic involution $\tau : \Gamma \to \Gamma$ which interchanges the points P_1 and P_2, and besides $\tau(k_1^{-1}) = \bar{k}_2^{-1}$ and $\tau(k_2^{-1}) = \bar{k}_1^{-1}$. Suppose that there exists a meromorphic 1–form Ω which has zeros at the points $\gamma_1, \dots, \gamma_g, \tau\gamma_1, \dots, \tau\gamma_g$ and simple poles at P_1 and P_2. Then e^u is a real–valued function.

Let M be a closed orientable surface and $f\colon M \to N$ be an immersion of M into a 3–dimensional Riemannian manifold N. The Willmore functional has the form:

$$\mathcal{W}(f) = \int_M \left(H^2 + \overline{K}\right) d\mu.$$

The functional $\mathcal{W}(f)$ is a conformal invariant.

In the 3–dimensional Euclidean space $\mathbb{E}^3$ the Willmore functional has the form $\mathcal{W}(f) = \int_M H^2 d\mu$. The minimizers of this functional are the round spheres.

The projection $E(k,\tau) \to E(k,\tau)/\mathrm{SO}(2)$ is a Riemannian submersion, where the factor–space $E(k,\tau)/\mathrm{SO}(2)$ has the metric:

$$d\tilde{s}^2 = \frac{1}{(1+\frac{k}{4}u^2)^2}du^2 + \frac{1}{1+\tau^2u^2}dv^2.$$

A profile of a rotationally invariant CMC surface satisfies:

$$\begin{cases} \dot{u} = \left(1+\frac{k}{4}u^2\right)\cos\sigma, \\ \dot{v} = \sqrt{1+\tau^2u^2}\sin\sigma, \\ \dot{\sigma} = 2H - \left(\frac{1}{u} - k\frac{u}{4}\right)\sin\sigma. \end{cases}$$

This system of ODE has the following first integral:

$$J = \frac{u}{1+\frac{k}{4}u^2}\left(\sin\sigma - Hu\right).$$

Theorem

The CMC spheres in $E(k,\tau)$ are critical points of the following Willmore–like functional:

$$E(f) = \mathcal{W}(f) + \int_M (-\frac{3}{4}\overline{K} + \frac{k}{4} - \frac{\tau^2}{4}) d\mu.$$

Theorem

For rotationally invariant spheres in $E(k,\tau)$ the functional $E(f)$ attains its minimum exactly at the CMC spheres.

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