Coarsening dynamics in inclusion processes and duality

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Outline

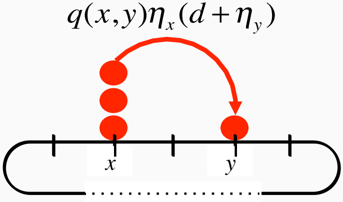
Introduction

Inclusion Processes : IP

Lattice: Λ of size $|\Lambda| = L$

State space: $E_{L,N} = \{0, 1, ..., N\}^{\Lambda}$

$$\boldsymbol{\eta} = (\eta_x : x \in \Lambda)$$



Jump rates: $q(x,y)\, c(\eta_x, \eta_y)$

complete graphs : $q(x,y) = \dfrac{1}{L-1}$ for all $x \neq y$

$c(k,l) = k(d+l)$ with parameter $d > 0$

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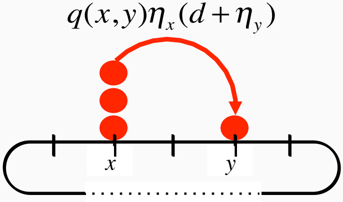

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Generator: $$\mathcal{L}h(\boldsymbol{\eta}) = \sum_{x,y \in \Lambda} \frac{1}{L-1} \eta_x (d + \eta_y) \big(h(\boldsymbol{\eta}^{x,y}) - h(\boldsymbol{\eta}) \big)$$

[Giardiá (2007) and (2010)]

Stationary Distributions

Product measures $\nu_\phi = \prod_{x \in \Lambda} \nu_\phi[\eta_x = n]$ with marginals

$$\nu_\phi[\eta_x = n] = \frac{1}{z(\phi)}\, \phi^n w(n) \quad \text{with} \quad w(n) = \frac{\Gamma(d+n)}{n!\Gamma(d)}$$

where $0 \leq \phi < \phi_c$ and Γ denotes the gamma function.

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The stationary weight

$$w(n) = \frac{\Gamma(d+n)}{n!\Gamma(d)} \sim n^{d-1} \text{ as } n \to \infty$$

The single site partition function is of the form

$$z(\phi) = \sum_{n=0}^{\infty} w(n)\phi^n = (1-\phi)^{-d}.$$

This implies that $z(\phi)$ diverges as $\phi \to 1$, hence the measures only exist for all $\phi \in [0, 1)$.

Condensation in homogeneous particle systems

canonical measures $\pi_{L,N}$ on $E_{L,N}$

spatial homogeneity $\pi_{L,N}[\eta_x \in .]$ do not depend on site x

density $\langle \eta_x \rangle_{L,N} := \sum_{n=1}^{N} n\, \pi_{L,N}[\eta_x = n] = N/L$

thermodynamic limit $L, N \to \infty\ , \quad N/L \to \rho \geq 0$

limiting single site marginal

$$\nu_\rho := \lim_{N/L \to \rho} \pi_{L,N}[\eta_x \in .]\ , \tag{1}$$

the first moment of the limiting distribution

$$\rho_b := \langle \eta_x \rangle_\rho \leq \rho\ . \tag{2}$$

Condensation

- A system with canonical distributions $\pi_{L,N}$ exhibits **condensation** in the thermodynamic limit $N/L \to \rho$ with background density ρ_b as in (2), if ν_ρ exists as defined in (1) and $\rho_b < \rho$.

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- A system with $\rho_b = 0$ is said to exhibit **complete condensation** if

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- If ν_ρ exists for all $\rho \geq 0$, the system exhibits a **condensation transition** with **critical density** $\rho_c \geq 0$ if

$$\rho_b \begin{cases} = \rho\ , & \text{for all } \rho < \rho_c \\ < \rho\ , & \text{for all } \rho > \rho_c \end{cases} . \qquad (4)$$

Dynamics of condensation

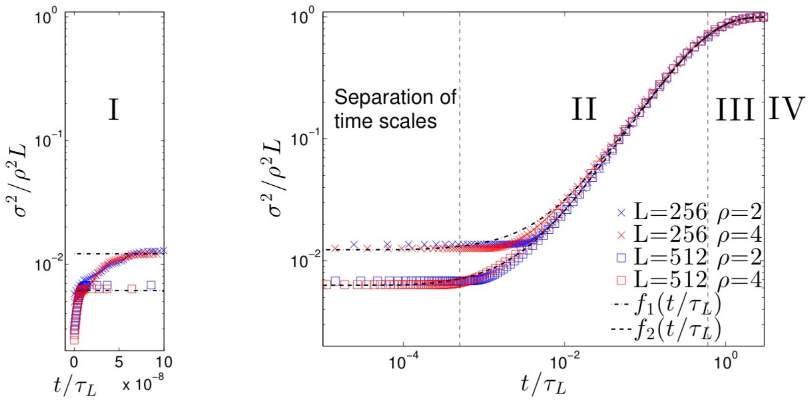

Figure 1: Different dynamical regimes in the TASIP.

[S.G., Cao (2014)]

Results

Mean-field equation

- For each $L, N \geq 1$, the process $\eta_x(t)$ is a birth death chain on the state space $E = \{0, 1, ..., N\}$ with

$$\beta_k = \frac{N-k}{L-1}(d+k) \text{ and } \mu_k = k\frac{(d(L-1)+N-k)}{L-1}.$$

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- $f_k^L(t) = \mathbb{E}[F_k(\eta(t))] = \frac{1}{L}\sum_{x\in\Lambda}\mathbb{P}[\eta_x(t) = k] \to f_k(t)$.
  Its master equation is

$$\frac{d}{dt}f_k(t) = (k+1)(d+\rho)f_{k+1}(t) + \rho(d+(k-1))f_{k-1}(t) - (dk+2\rho k+\rho d)f_k(t), \quad (5)$$

  valid for all $k \geq 0$ with the convention $f_{-1}(t) \equiv 0$ for all $t \geq 0$.

Analysis of the single site limit dynamics

Denote the limit process with the master equation (5) by $(Y_t : t \geq 0)$. Its generator is in the form

$$\mathcal{L}_{\mathrm{BD}} h(k) = k(d+\rho)(h(k-1) - h(k)) + \rho(d+k)(h(k+1) - h(k)). \quad (6)$$

Try $h(k) = k$:

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Try $h(k) = k^2$:

Analysis of the size-biased limiting dynamics

Size-biased dynamics $(X_t : t \geq 0)$ on state space $\mathbb{N}$:

$$p_k(t) = \frac{k}{\rho} f_k(t),\ k \geq 1 \text{ and } t \geq 0$$

For $k = 1$,

$$\frac{d}{dt}p_1(t) = (d+\rho)p_2(t) - 2\rho(d+1)p_1(t) + \left(\sum_{k\geq 2}[d - \frac{1}{k}d\rho]p_k(t)\right).$$

For $k \geq 2$,

$$\begin{aligned}\frac{d}{dt}p_k(t) &= k(d+\rho)p_{k+1}(t) + \frac{k}{k-1}\rho(d+(k-1))p_{k-1}(t) \\ &\quad - \left[(k-1)(d+\rho) + \frac{k+1}{k}\rho(d+k)\right]p_k(t) \\ &\quad + \left[\frac{1}{k}d\rho - d\right]p_k(t).\end{aligned}$$

Analysis of the size-biased limiting dynamics

A birth death chain with killing/cloning with rates

$$
\begin{aligned}
&\text{birth rate} && \frac{k+1}{k}\rho(d+k), \;\; \text{for } k>0 \;, \\
&\text{death rate} && (k-1)(d+\rho), \;\; \text{for } k>1 \;, \\
&\text{rate from } k \text{ to } 1 && \left(d-\frac{1}{k}d\rho\right)_+, \;\; \text{for } k>1 \;, \\
&\text{cloning rate} && \left(\frac{1}{k}d\rho-d\right)_+, \;\; \text{for } k>1 \;, \\
&\text{killing rate} && \sum_{k>1}\left(\frac{1}{k}d\rho-d\right)_+, \;\; \text{for } k=1 \;, \qquad (8)
\end{aligned}
$$

where we again denote by $(\cdot)_+ = \max\{0,(\cdot)\}$ the positive part of the expression.

Analysis of the size-biased limiting dynamics

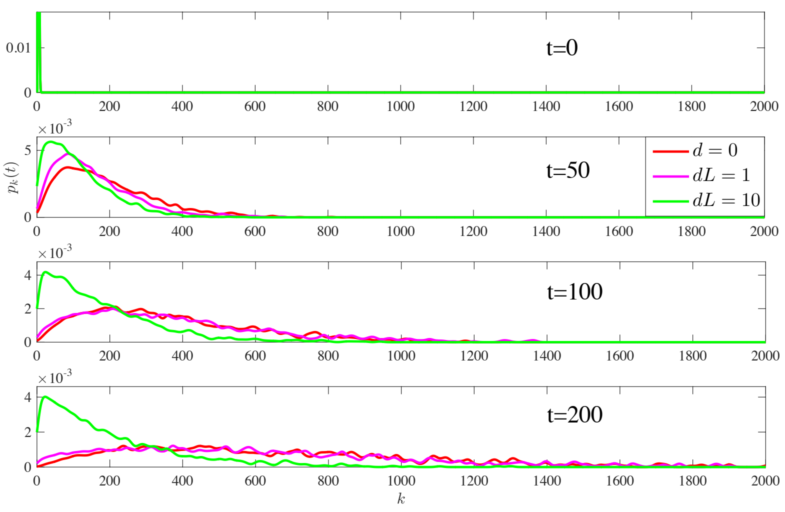

Figure 2: $p_k(t)$ of system size $L = 1024$ for $\rho = 2$ from the simulation of inclusion processes on complete graph with $d = 0$ (red) and $dL = 1$ (magenta) and $dL = 10$ (blue) at time 10, 50, 100 and 200.

Case $d = 0$: Single site limit dynamics

For $d = 0$, $(Y_t : t \geq 0)$ is a standard birth death process with the same birth and death rates $\beta_k = \mu_k = \rho k$.
This could be solved

1. directly using the "Generating function",

$$G(z,t) = \sum_{k=0}^{\infty} z^k f_k(t) = \mathbb{E}[z^{Y_t} \mid Y_0],$$

   conditional on the initial state Y_0.

2. using our method by analysing the master equation

$$\frac{d}{dt} f_k(t) = \rho(k+1) f_{k+1}(t) + \rho(k-1) f_{k-1}(t) - 2\rho k f_k(t), \qquad (9)$$

The solution of (9) separates into a bulk and condensed part

$$f_k(t) = f_k^{\text{bulk}}(t) + f_k^{\text{cond}}(t). \tag{10}$$

- $f_k^{\text{bulk}}(t) = f_0(t)\delta_{k,0} \to \delta_{k,0}$ as $t \to \infty$
- $f_k^{\text{cond}}(t) = \epsilon_t^2 h(u), \quad$ with $u = k\epsilon_t$ leading to

$$\frac{\dot{\epsilon}_t}{\epsilon_t}[uh'(u) + 2h(u)] == 2\rho\epsilon_t h'(u) + \rho u \epsilon_t h''(u).$$

With $\epsilon_t = \frac{1}{\rho t}$ and thus $\frac{\dot{\epsilon}_t}{\epsilon_t^2} = -\rho$, we have

$$uh''(u) + (2+u)h'(u) + 2h(u) = 0, \tag{11}$$

which has the solution $h(u) = \rho e^{-u}$.

Case $d = 0$: Size-biased limiting dynamics

Size-biased dynamics $(X_t : t \geq 0)$ on state space $\mathbb{N}$:

$$p_k(t) = \frac{k}{\rho} f_k(t), \ k \geq 1 \text{ and } t \geq 0$$

For the case $d = 0$, we have the size-biased birth death chain $(X_t : t \geq 0)$ on $E = \mathbb{N}$ with birth and death rates

$$\beta_k = \rho(k+1) \text{ and } \mu_k = \rho(k-1), \tag{12}$$

corresponding to the master equation

$$\frac{d}{dt} p_k(t) = \rho k p_{k+1}(t) + \rho k p_{k-1}(t) - 2\rho k p_k(t), \tag{13}$$

for all $k \geq 1$ with the convention $p_0(t) = 0$.

Analysis of the size-biased limiting dynamics $d = 0$

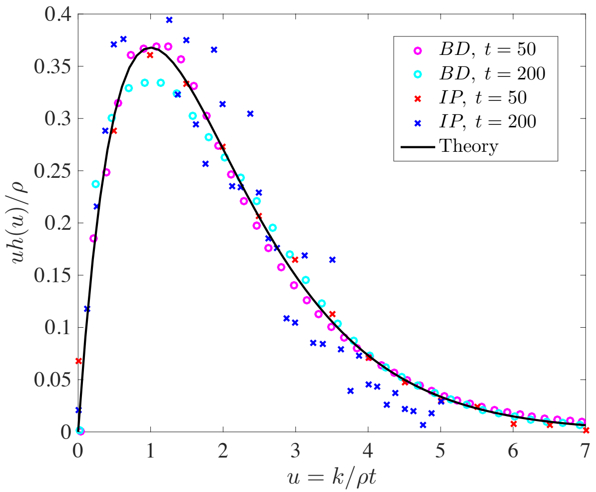

Figure 3: Normalised theoretical prediction $uh(u)$ plotted against the rescaled variable $u = \frac{1}{\rho t}$. Data from direct IP simulations with averaging over 500 realisations of $m = L = 1024$ copies with $d = 0,\ \rho = 2$.

Duality

Suppose $\eta = (\eta(t) : t \geq 0)$ and $\xi = (\xi(t) : t \geq 0)$ are two Markov processes with state spaces E_1 and E_2, respectively.
The process η is said to be dual to process ξ with respect to the duality function $D \in C(E_1 \times E_2)$, if

$$\mathbb{E}^\eta D(\eta(t), \xi) = \mathbb{E}^\xi D(\eta, \xi(t)), \tag{14}$$

for all $\eta \in E_1, \xi \in E_2$ and $t \geq 0$.

Self-Duality

Let $\eta = (\eta(t) : t \geq 0)$ and $\xi = (\xi(t) : t \geq 0)$ be two copies of the same Markov processes on a state space E.
The process is said to be self-dual with self-duality function $D : E \times E \to \mathbb{R}$ if for all $\eta \in E$ and $\xi \in E$, (14) holds.

Self-dual property of the inclusion process

The symmetric inclusion process is self-dual with the duality function

$$D(\xi,\eta) = \prod_x d(\xi_x,\eta_x) \quad . \tag{15}$$

where

$$d(k,n) = \frac{n!}{(n-k)!}\frac{\Gamma(d)}{\Gamma(d+k)}.$$

The self-duality is then given by

$$\mathbb{E}^{\eta}[D(\xi,\eta(t))] = \mathbb{E}^{\xi}[D(\xi(t),\eta]. \tag{16}$$

[Giardiá (2009) and (2010)]

Analysis using duality : time dependent variances

- For a positive $d = d_L \to 0$, the analysis is not accessible with previous methods.
- Use the self duality property of the IP : a dual process containing only two particles.
- Compute second moment of the occupation numbers for fixed system size of the process :

$$\begin{aligned} C^L_{xx}(t) &:= \mathbb{E}^{\nu_\rho}[\eta_x^2(t)] \\ &= \sigma^2(0)\mathbb{P}^{x,x}[X_t = Y_t] + \left(\frac{d\rho(1+\rho)+\rho^2}{d}\right)\mathbb{P}^{x,x}[X_t \neq Y_t], \end{aligned}$$

where X_t and Y_t denote the particle positions for an inclusion process with two particles on the lattice Λ, and $\mathbb{P}^{x,y}$ the path measure with initial values $X_0 = x, Y_0 = y$.

Analysis using duality : exact compuations for two dual particles

In the complete graph case, the process $Z_t = \mid Y_t - X_t \mid$ takes values on the state space $\{0, 1\}$ with the dynamics :

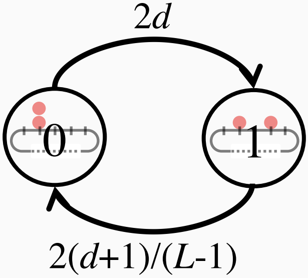

Figure 4: Two dual particle system of the inclusion process.

Analysis using duality : exact compuations for two dual particles

- The Q matrix is diagonalisable : $Q = U\Lambda U^{-1}$
-

$$P_t = e^{tQ} = U \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{2(1+dL)}{L-1}t} \end{pmatrix} U^{-1}$$

$$= \frac{1}{(1+dL)} \begin{pmatrix} (d+1) + d(L-1)e^{-\frac{2(1+dL)}{L-1}t} & d(L-1)[1 - e^{-\frac{2(1+dL)}{L-1}t}] \\ (d+1)[1 - e^{-\frac{2(1+dL)}{L-1}t}] & d(L-1) + 2(d+1)e^{-\frac{2(1+dL)}{L-1}t} \end{pmatrix}.$$

- As $t \to \infty$, we have

$$P_t \to \begin{pmatrix} \alpha & 1-\alpha \\ \alpha & 1-\alpha \end{pmatrix},$$

where $\alpha = \frac{d+1}{dL+1}$.

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- As $t \to \infty$, we have
$$P_t \to \begin{pmatrix} \alpha & 1-\alpha \\ \alpha & 1-\alpha \end{pmatrix},$$
where $\alpha = \frac{d+1}{dL+1}$. Hence,
$$\mathbb{P}(Z_t = i) \to \begin{cases} \alpha & \text{if } i = 0, \\ 1-\alpha & \text{if } i = 1, \end{cases} \tag{17}$$

Analysis using duality : exact compuations for two dual particles

$$C^L_{xx}(t) = \sigma^2(0)\mathbb{P}^0[Z_t = 0] + \left(\frac{d\rho(1+\rho)+\rho^2}{d}\right)\mathbb{P}^0[Z_t = 1]$$
$$= C^L_{xx}(t) = \sigma^2(0) + \frac{\rho^2(L-1)}{1+dL}(1 - e^{-\frac{2(1+dL)}{L-1}t})$$

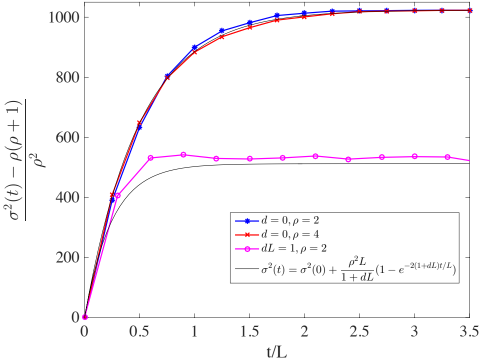

Figure 5: $\sigma^2(t)$ from simulation data for $L = 1024$, over 500 realisations.

Summary

Summary

Results so far

- rigorously derive the mean field master equation for IP
- scaling analysis of the sigle site dynamics
- size-biased BD chains for IP provide efficient sampling tool.

Work in progress

- rigorously establish scaling solution and coarsening scaling law for IP
- generalised version of size-biased BD chain
- sampling tool for explosive condensation models with
  $c(k,l) = k^{\gamma}(d + l^{\gamma})\ , \quad \gamma \geq 1$
- apply this model to evolution process.

Thank you!