Recurrence relations for the sections of the generating series of the solution to the multidimensional difference equation

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Generating functions (or z-transformation) are a powerful tool in enumerative combinatorial analysis, and the study of their properties is of great interest, especially when it comes to a function belonging to one of the classes in the hierarchy proposed by Richard Stanley (1990):

$$\text{rational} \subset \text{algebraic} \subset \text{D-finite}.$$

Stanley noted that rational generating functions are "the most useful" class of generating functions.
A. De Moivre (1722) considered power series

$$F(z) = a_0 + a_1 z + ... + a_k z^k + ...$$

with coefficients $a_0, a_1, ..., a_k, ...$, forming a recursive sequence, i.e. a sequence satisfying a relation

$$c_0 a_{m+p} + c_1 a_{m+p-1} + ... + c_m a_p = 0,\ p = 0, 1, 2...,$$

where $c_j, j = 0, \ldots, p$, are some constants. It turned out that such series always represent rational functions.

$n = 1$ Fibonacci numbers

The most famous case where we encounter a difference equation is the Fibonacci numbers:

$$f(1) = 1,$$
$$f(2) = 1,$$
$$f(x) = f(x-1) + f(x-2), x \geqslant 2$$

A solution to this equation is well-known:

$$f(x) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^x - \left(\frac{1-\sqrt{5}}{2} \right)^x \right].$$

The generating function for $f(x)$ is

$$F(z) = \frac{z^2 f(0) + z(f(1) + f(0))}{z^2 - z - 1}.$$

More precisely, the following assertion is true.

Theorem (Moivre, 1722)

The power series $F(z)$ is recursive if and only if it represents rational function.

The proof of this simple fact can be found, for example, in [Stanley, 1990]. An analogue of the Moivre theorem for a multidimensional difference equation with constant coefficients was proved by E. Leinartas and A. L. in 2009: the generating function of the solution of a multidimensional difference equation with constant coefficients is rational if and only if the generating function of the initial data is rational.

D-finite power series

Let $F(z) = \sum\limits_{x \in \mathbb{Z}^n_{\geqslant}}$ be a formal power series of $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.
Series $F(z)$ is called D-finite if it satisfies to a system of differential equations

$$\begin{cases} P_k^1(z)\frac{\partial^k F}{\partial z_1^k} + P_{k-1}^1(z)\frac{\partial^{k-1} F}{\partial z_1^{k-1}} + \cdots + P_1^1(z)\frac{\partial^1 F}{\partial z_1} + P_0^1(z)F = 0, \\ P_k^2(z)\frac{\partial^k F}{\partial z_2^k} + P_{k-1}^2(z)\frac{\partial^{k-1} F}{\partial z_2^{k-1}} + \cdots + P_1^2(z)\frac{\partial^1 F}{\partial z_2} + P_0^2(z)F = 0, \\ \ldots \\ P_k^n(z)\frac{\partial^k F}{\partial z_n^k} + P_{k-1}^n(z)\frac{\partial^{k-1} F}{\partial z_n^{k-1}} + \cdots + P_1^n(z)\frac{\partial^1 F}{\partial z_n} + P_0^n(z)F = 0, \end{cases}$$

where P_j^i are polynomials.
An analogue of Moivre theorem for D-finite series was derived by T. Yakovleva (Nekrasova) in 2014.

In this research we find recurrence relations for sections of generating series of solutions of multidimensional difference equations and prove an analog of the Moivre theorem for sections of such generating series.

Let $\mathbb{Z}$ and $\mathbb{Z}_{\geqslant}$ be the set of integers and non-negative integers, and $\mathbb{Z}^n = \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{n \text{ times}}$ and $\mathbb{Z}^n_{\geqslant} = \underbrace{\mathbb{Z}_{\geqslant} \times \ldots \times \mathbb{Z}_{\geqslant}}_{n \text{ times}}$. For a set $\Delta = \{\alpha^0, \alpha^1, \ldots, \alpha^N\} \subset \mathbb{Z}^n_{\geqslant}$ of $N+1$ vectors with non-negative coordinates $\alpha^j = (\alpha^j_1, \ldots, \alpha^j_n)$ for $j = 0, \ldots, N$ and $c_0, c_1, \ldots, c_N \in \mathbb{C}$, we define a linear difference equation with constants coefficients

$$\sum_{j=0}^{N} c_j f(x - \alpha^j) = 0, \quad x \geqslant m,$$

where $\alpha^0 = (0, \ldots, 0)$, $c_0 = 1$, $m = (m_1, \ldots, m_n)$, $m_i = \max\limits_{1 \leqslant j \leqslant N} \alpha^j_i$, for $i = 1, \ldots, n$, and the inequality $x \geqslant m$ means that $x_i \geqslant m_i$ for all $i = 1, \ldots, n$.

The characteristic polynomial of this equation is $P(z) = \sum\limits_{j=0}^{N} c_j z^{-\alpha^j}$.

We formulate the *Cauchy problem*: find a function $f(x)$ that satisfies the equation

$$\sum_{j=0}^{N} c_j f(x-\alpha^j)=0, \quad x \geqslant m, \tag{1}$$

and coincides with some given initial data function $\varphi : \mathbb{Z}^n \to \mathbb{C}$ on $X_0=\{x \in \mathbb{Z}^n_{\geqslant} : x \ngeqslant m\} = \mathbb{Z}^n_{\geqslant} \setminus \left(m+\mathbb{Z}^n_{\geqslant}\right)$:

$$f(x)=\varphi(x), \;\; x \ngeqslant m. \tag{2}$$

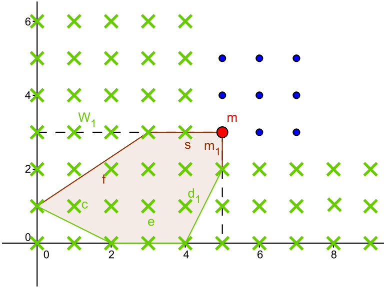
Multidimensional difference equations of this kind arise in the theory of digital recursive filters, as well as in combinatorial analysis, where they are called linear recursive relations.

Example 1

For a difference equation

$$f(x_1,x_2) = f(x_1-2,x_2)+f(x_1-5,x_2-2)+f(x_1-3,x_2-3)+ \\ +f(x_1-1,x_2-3)+f(x_1,x_2-2)$$

the set of initial data $(x_1,x_2)\not\geqslant(5,3)$ is infinite:



Example 2. Bloom's strings

Bloom studies the number of singles in x-length bit strings, where a single is any isolated 1 or 0, i.e., any run of length 1. Let $r(x, y)$ be the number of x-length bit strings beginning with 0 and having y singles.

For $x = 5$ we get:

- $r(5, 0) = 3$, since 00000, 00011, 00111
- $f(5, 1) = 5$, since 00001, 00100, 00110, 01100, 01111
- $f(5, 2) = 3$, since 00010, 01000, 01110
- $f(5, 3) = 4$, since 00101, 01001, 01011, 01101
- $f(5, 4) = 0$,
- $f(5, 5) = 1$, since 01010

Example 2

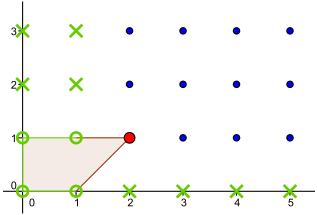
Bloom's numbers $r(x,y)$ satisfy the difference equation

$$r(x,y) = r(x-1,y) + r(x-1,y-1) + r(x-2,y) - r(x-2,y-1).$$

with the "initial data"

$$\varphi(0,0) = 1, \varphi(1,0) = 0; \varphi(x,0) = \varphi(x-1,0) + \varphi(x-2,0), x \geqslant 2,$$

$$\varphi(1,1) = 1, \varphi(0,y) = 0, y \geqslant 1;\ \varphi(1,y) = 0, y \geqslant 2.$$



For the function $f : \mathbb{Z}^n \to \mathbb{C}$, we define *generating series*

$$F(z) = \sum_{x \in \mathbb{Z}^n_{\geqslant}} f(x)\, z^x,$$

and its sections as follows.
For $j = 1, \ldots, N$ we define a projection operator $\pi_j : \mathbb{Z}^n \to \mathbb{Z}^n$ as

$$\pi_j : (x_1, \ldots, x_n) \to (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n).$$

Fix a set $J = \{j_1, \ldots, j_k\} \subset \{1, 2, \ldots, n\}$, $1 \leqslant j_1 < \ldots < j_k \leqslant n$ and denote by π_J the superposition of such operators:

$$\pi_J = \pi_{j_1} \circ \pi_{j_2} \circ \cdots \circ \pi_{j_k}.$$

Let $J' = \{1, 2, \ldots, N\} \setminus J$, and $x_J = \pi_J x, x_{J'} = \pi_{J'} x$, which means that coordinates with indices from J and J' are zeros.

Then the generating series $F(z)$ can be represent as follows

$$F(z) = \sum_{x_J \in \mathbb{Z}^n_{\geqslant}} \left(\sum_{x_{J'} \in \mathbb{Z}^n_{\geqslant}} f(x)\, z^{x_{J'}} \right) z^{x_J} = \sum_{x_J \in \mathbb{Z}^n_{\geqslant}} F(x_J; z_{J'}) z^{x_J},$$

and the series

$$F(x_J; z_{J'}) = \sum_{x_{J'} \in \mathbb{Z}^n_{\geqslant}} f(x_{J'})\, z^{x_{J'}}, \ \ x_J \in \mathbb{Z}^n_{\geqslant}$$

is naturally called *sections* of the generating series $F(z)$.

Note that if $J = \{1, 2, \ldots, n\}$, then $J' = \varnothing$, $x_J = x$, and $F(x_J; z_{J'}) = f(x)$ is a function from difference equation (1), and if $J = \varnothing$, $J' = \{1, 2, \ldots, n\}$, then $F(x_J; z_{J'}) = F(z)$ is a generating series.

We consider the function $\widetilde{\varphi}(x) : \mathbb{Z}^n_{\geqslant} \to \mathbb{C}$, which coincides with the initial data function $\varphi(x)$ on the set X_0 and vanishes on its complement $\mathbb{Z}^n_{\geqslant} \setminus X_0$, and define series

$$\Phi(x_J; \tau_{J'}; z_{J'}) = \sum_{\substack{x_{J'} \in \mathbb{Z}^n_{\geqslant} \\ x_{J'} \ngeqslant \tau_{J'}}} \widetilde{\varphi}(x)\, z^{x_{J'}},$$

and *generating series* of initial data

$$\Phi(z) = \Phi(x_\varnothing; m; z) = \sum_{x \in \mathbb{Z}^n_{\geqslant}} \widetilde{\varphi}(x)\, z^x$$

and its *sections*

$$\Phi(x_J; z_{J'}) = \Phi(x_J; m; z_{J'}) = \sum_{x_{J'} \in \mathbb{Z}^n_{\geqslant}} \widetilde{\varphi}(x_{J'})\, z^{x_{J'}}.$$

The sections of the generating series for solving the multidimensional difference equation are interconnected by recurrent relations.

Theorem (S. Akhtamova, A. L.)

The sections of the generating series of the solution of the Cauchy problem (1)–(2) *satisfy the recursive relation*

$$\sum_{j=0}^{N} c_j z^{\alpha^j_{J'}} F(x_J - \alpha^j_J; z_{J'}) = \sum_{j=0}^{N} c_j z^{\alpha^j_{J'}} \Phi(x_J - \alpha^j_J; m_{J'} - \alpha^j_{J'}; z_{J'}), \quad (3)$$

for all $x_J \geqslant m_J$.

To prove the theorem we fix some subset J and multiply both sides of equation (1) by $z^{x_{J'}}$ and sum over all integers $x_{J'} \geqslant m_{J'} = \pi_{J'} m$.

We consider special cases of the formula (3) with $J' = \varnothing$ and $J' = M$. For $J' = \varnothing$ the identity takes the form of the original difference equation, since neither multiplication by $z_{J'}$ nor summation over $x_{J'}$ takes place.
The case $J = \varnothing, J' = M$ was studied in detail by E. Leinartas and A. L. in 2009:

$$\underbrace{\sum_{j=0}^{N} c_j z^{\alpha_j}}_{=P(\frac{1}{z})} F(z) = \sum_{j=0}^{N} c_j z^{\alpha^j} \Phi(x_{\varnothing}; m - \alpha^j; z), \quad (4)$$

where $P(z)$ is the characteristic polynomial for (1). Using this formula in [**?**], the generating function of the solution of the difference equation is calculated in terms of the generating function of the initial data and the coefficients of the difference equation, and then a multidimensional analog of the Moivre theorem was proved.

Now we can formulate an analog of the Moivre theorem for sections of the generating series of solutions to a multidimensional difference equation, the proof of which is a direct consequence of Theorem 1.

Theorem (S. Akhtamova, A. L.)

The sections $F(x_J; z_{J'})$ of the generating series $F(z)$ are rational if and only if the sections of the generating series of the initial data $\Phi(x_J; z_{J'})$ are rational.

The most natural class of problems related to difference equations of the form (1) are problems on the integer lattice paths. The best-known classes of such paths are the Dick, Motzkin, and Schroeder paths.

Let set $\Delta = \{\alpha^1, \ldots, \alpha^N\}$ consist of N non-zero vectors with non-negative integer coordinates $\alpha^j = (\alpha_1^j, \ldots, \alpha_n^j) \in \mathbb{Z}^n_{\geqslant}$, for all $j = 1, \ldots, N$, and $\alpha^0 = (0, \ldots, 0)$. We assume that all paths begin at the origin, and the (solid) cone K spanned by vectors from the set Δ is pointed (that is, it does not contain any straight line). Then the number of paths from the origin to the point $x \in \mathbb{Z}^n$ is finite and satisfies the equation (1), and the initial data $\varphi(x), x \not\geqslant m$ for the Cauchy problem also satisfy the difference equation and can be recovered from the condition that $\varphi(x) = 0, x \notin \mathbb{Z}^n_{\geqslant}$ and $\varphi(0) = 1$.

Theorem

Sections of lattice paths with steps from set Δ satisfy the recurrence relation (difference equation)

$$\sum_{j=0}^{N} c_j z^{\alpha^j_{J'}} F(x_J - \alpha^j_J; z_{J'}) = 0$$

for all $x_J \geqslant m_J$.

We note that the sections $F(x^0_J; z_{J'})$ are the generating functions for the number of paths with steps from Δ from the origin to all points of the plane $x \in \mathbb{Z}^n : x_J = x^0_J$.

Example 3

Let $\alpha^1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \alpha^2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, c_1 = c_2 = -1$, then $m = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and the difference equation is

$$f(x_1, x_2) - f(x_1 - 1, x_2 - 2) - f(x_1 - 3, x_2 - 1) = 0, \qquad (5)$$

for $(x_1, x_2) \geqslant (3, 2)$, and the initial data are given as

$$f(x_1, x_2) = \varphi(x_1, x_2)$$

on the set $(x_1, x_2) \not\geqslant (3, 2)$.

The case $J' = \varnothing$ is of no interest.

Example 3

In the case of $J' = \{1\}$ and $J = \{2\}$, the sections of the generating function are related by a formulae

$$F(x_2; z_1) - z_1 F(x_2 - 2; z_1) - z_1^3 F(x_2 - 1; z_1) =$$

$$= \Phi(x_2; 3; z_1) - z_1 \Phi(x_2 - 2; 2; z_1) - z_1^3 \Phi(x_2 - 1; 0; z_1), x_2 \geqslant 2,$$

and the expressions on the right side are finite sums (which is true only for the case $n = 2$) of the form

$$\Phi(x_2; 3; z_1) = \varphi(0, x_2) + \varphi(1, x_2) z_1 + \varphi(2, x_2) z_1^2,$$

$$\Phi(x_2 - 2; 2; z_1) = \varphi(0, x_2 - 2) + \varphi(1, x_2 - 2) z_1,$$

$$\Phi(x_2 - 2; 0; z_1) = 0.$$

The "initial data" of this (one-dimensional) Cauchy problem are the series

$$F(0; z_1) = \Phi(0; z_1) \text{ and } F(1; z_1) = \Phi(1; z_1).$$

Example 3

In the case of $J' = \{2\}$ and $J = \{1\}$, the sections of the generating function are related by a formulae

$$F(x_1; z_2) - z_2^2 F(x_1 - 1; z_2) - z_2 F(x_1 - 3; z_2) =$$
$$= \Phi(x_1; 2; z_2) - z_1 \Phi(x_1 - 1; 0; z_2) - z_1^3 \Phi(x_1 - 3; 1; z_2), x_1 \geqslant 3,$$

and the expressions on the right side are

$$\Phi(x_1; 2; z_2) = \varphi(x_1, 0) + \varphi(x_1, 1) z_2,$$

$$\Phi(x_1 - 1; 0; z_2) = 0, \Phi(x_1 - 3; 1; z_2) = \varphi(x_1 - 3, 0).$$

The "initial data" of this Cauchy problem are

$$F(0; z_2) = \Phi(0; z_2), F(1; z_2) = \Phi(1; z_2) \text{ and } F(2; z_2) = \Phi(2; z_2).$$

With an appropriate choice of set Δ of steps, the sections of the generating series for the number of paths on the integer lattice will represent known sequences of polynomials.

Example 4. Fibonacci polynomials

For a set of steps $\alpha^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha^2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ the number of paths satisfies equation

$$f(x_1, x_2) - f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2) = 0,$$

then the sections of the generating series for the function $f(x_1, x_2)$ satisfy the recursive relation

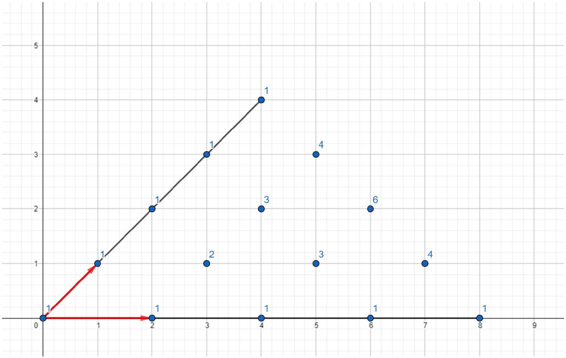
$$F(x_1, z_2) - z_2 F(x_1 - 1, z_2) - F(x_1 - 2, z_2) = 0$$

with initial data $F(0; z_2) = 1, F(1; z_2) = z_2$. Continuing the calculations yields the known sequence of Fibonacci polynomials:

$$\begin{aligned} F(2; z_2) &= z_2^2 + 1, \\ F(3; z_2) &= z_2^3 + 2z_2, \\ F(4; z_2) &= z_2^4 + 3z_2^2 + 1, \\ &\dots \end{aligned}$$

Example 4. Fibonacci polynomials

$F(0; z_2) = 1, F(1; z_2) = z_2,\ F(2; z_2) = z_2^2 + 1,\ F(3; z_2) = z_2^3 + 2z_2,$
$F(4; z_2) = z_2^4 + 3z_2^2 + 1,\ F(5; z_2) = z_2^5 + 4z_2^3 + 3z_1,\ \ldots$



Example 5. Pell polynomials

For a set of steps $\alpha^1 = \alpha^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha^3 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (we assume that the step α^1 has multiplicity two or that there are two such steps of different colors) the function $f(x_1, x_2)$ of the number of paths from the origin to the point (x_1, x_2) satisfies the difference equation

$$f(x_1, x_2) - 2f(x_1 - 1, x_2 - 1) - f(x_1 - 2, x_2) = 0.$$

The sections $F(x_1; z_2)$ satisfy the recursive relation

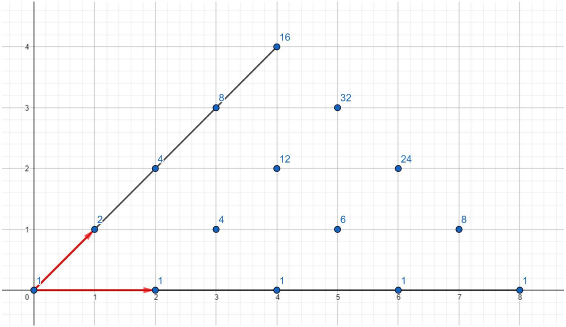
$$F(x_1; z_2) - 2z_2F(x_1 - 1; z_2) - F(x_1 - 2; z_2) = 0$$

with initial data $F(0; z_2) = 1, F(1; z_2) = 2z_2$. Continuing the calculations, we obtain the known sequence of Pell polynomials:

$$\begin{aligned} F(2; z_2) =& 4z_2^2 + 1, \\ F(3; z_2) =& 8z_2^3 + 4z_2, \\ F(4; z_2) =& 16z_2^4 + 12z_2^2 + 1, \\ & \dots \end{aligned}$$

Example 5. Pell polynomials

$$F(0;z_2)=1, F(1;z_2)=2z_2,\ F(2;z_2)=4z_2^2+1,\ F(3;z_2)=8z_2^3+4z_2,$$
$$F(4;z_2)=16z_2^4+12z_2^2+1,\ F(5;z_2)=32z_2^5+32z_2^3+6z_2,\ \ldots$$



Futher generalizations

We consider **a special case**, when $\Delta = \{e^1, e^2, \ldots, e^N\} \subset \mathbb{Z}^N$ is a set of unit vectors in the standard basis, $e^j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is only on the j-th position:

$$c_0 f(\lambda) + c_1 f(\lambda - e^1) + \ldots + c_N f(\lambda - e^N) = g(\lambda), \qquad (6)$$

for $\lambda \in \mathbb{Z}^N_{\geqslant} + e^1 + \ldots + e^N$ and the initial data function is

$$f(\lambda) = \varphi(\lambda).$$

for $\lambda \ngeqslant m = e^1 + \ldots + e^N$.

Let cones K_p and L_p be spanned by vectors from $\Delta_1 = \{e^1, \ldots, e^p\}$ and $\Delta_2 = \{e^{p+1}, \ldots, e^N\}$ respectively. Since vectors in set Δ are linearly independent, each element λ of cone K can be represented as a unique sum of elements x and y from cones K_p and L_p:

$$K \ni \lambda = x + y, y \in K_p, x \in L_p.$$

Consequently, generating series $F(z)$ can be represented as a sum

$$F(z) = \sum_{x \in L_p} F(K_p; x; z)\, z^x, \text{ where } F(K_p; x; z) = \sum_{y \in K_p} f(x+y) z^y.$$

We call $F(K_p; x; z)$ as **a section** of generating series $F(z)$.

Let $\pi_j : \mathbb{Z}^N \to \mathbb{Z}^N$ is a projection operator defined as:

$$\pi_j : (x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_N) \mapsto (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_N).$$

Let $\delta_j : \mathbb{Z}^N \to \mathbb{Z}^N$ is a shift operator $\delta_j : x \mapsto x - e^j$ or:

$$\delta_j : (x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_N) \mapsto (x_1, \ldots, x_{j-1}, x_j - 1, x_{j+1}, \ldots, x_N).$$

Lemma (1)

If $F(K_p;x;z) = \sum\limits_{y\in K_p} f(x+y)z^y$, *then*

$$(1-\pi_j)F(K_p;x;z) = z_j F(K_p;x+e^j;z)$$

for $1 \leqslant j \leqslant p$.

Proof.

$$(1-\pi_j)F(K_p;x;z) = \sum_{y\in K_p+e^j} f(x+y)z^y =$$

$$= z_j \sum_{y\in K_p} f(x+y+e^j)z^y = z_j F(K_p;x+e^j;z)$$

for $1 \leqslant j \leqslant p$.

Lemma (2)

$$(1-\pi_1)z_1F(K_1;x;z) = (1-\pi_1)\delta_1F(K_1;x;z)$$

.

Proof.

$$(1-\pi_1)z_1F(K_1;x;z) = z_1F(K_1;x;z) = z_1\sum_{y\in K_1} f(x+y)z^y =$$

$$= \sum_{y\in K_1} f(x+y)z^{y+e^1} = \sum_{y\in e^1+K_1} f(x+y-e^1)z^y =$$

$$= \sum_{y\in K_1} f(x+y-e^1)z^y - f(x-e^1) =$$

$$= F(K_1;x-e^1;z) - F(K_1;x-e^1;0) = (1-\pi_1)F(K_1;x-e^1;z) =$$

$$= (1-\pi_1)\delta_1F(K_1;x;z).$$

Lemma (3)

$\pi_p F(K_p; x; z) = F(K_{p-1}; x; z)$ *for* $1 \leqslant p \leqslant N$.

Proof. Since

$$\pi_p \sum_{y \in K_p} f(x+y) z^y = \sum_{y \in K_{p-1}} f(x+y) z^y.$$

Lemma (4)

$(1 - \pi_j) G(K_p; x; z) = G(K_p; x; z)$ *for* $1 \leqslant j \leqslant p$.

Proof. Since initial data function $g(x)$ is given on $x \geqslant I$, we acknowledge that $g(x) = 0$ for $x \ngeqslant I$, which yields

$$\pi_j G(K_p; x; z) = 0$$

for $1 \leqslant j \leqslant p$.

Theorem

Let $\Pi_p = (1-\pi_1)\dots(1-\pi_p)$ *and*

$$P(z_1,\dots,z_p,\delta_{p+1},\dots,\delta_N) = 1+\dots+c_p z_p + c_{p+1}\delta_{p+1}+\dots+c_N z_N.$$

The sections $F(K_p;x;z)$ *of generating series* $F(z)$ *satisfied to the recurrence relation*

$$\Pi_p P(z_1,\dots,z_p,\delta_{p+1},\dots,\delta_N) F(K_p;x;z) = \Pi_p G(K_p;x;z). \qquad (7)$$

Proof. By Lemma 1 and 2 we get

$$\Pi_p P(z_1,\dots,z_p,\delta_{p+1},\dots,\delta_N) F(K_p;x;z) =$$
$$= \Pi_p P(\delta_1,\dots,\delta_p,\delta_{p+1},\dots,\delta_N)\cdot \sum_{y\in K_p} f(x+y)z^y = \sum_{y\in K_p+I_p} P(\delta) f(x+y) z^y$$

and

$$\Pi_p G(K_p;x;z) = G(K_p;x;z) = \sum_{y\in K_p+I_p} g(x+y)z^y,$$

then equalling coefficients by z^y yields (6), $I_p = e_1+\dots+e_p$.

Example 6

Let $N = 3$ and consider a difference equation

$$c_0 f(\lambda) + c_1 f(\lambda - e^1) + c_2 f(\lambda - e^2) + c_3 f(\lambda - e^3) = g(\lambda), \lambda \geqslant I.$$

For $p = 0$ cone K_0 is an empty set and for $x \in L_0 + I$ we get the given difference equation

$$c_0 F(K_0; x; z) + c_1 F(K_0; x - e^1; z) + c_2 F(K_0; x - e^2; z) + \\ + c_3 F(K_0; x - e^3; z) = G(K_0; x; z),$$

where $F(K_0; x; z) = f(x)$ and $G(K_0; x; z) = g(x)$.

For $p = 1$ we get that cone $K_1 = \langle e^1 \rangle$ and for $x \in L_1 + e^2 + e^3$ we get

$$(1 - \pi_1)(c_0 + c_1 z_1 + c_2 \delta_2 + c_3 \delta_3) F(K_1; x; z) = (1 - \pi_1) G(K_1; x; z),$$

and

$$(c_0 + c_1 z_1) F(K_1; x; z) + c_2 F(K_1; x - e^2; z) + c_3 F(K_1; x - e^3; z) =$$
$$= G(K_1; x; z) + c_0 F(K_1; x; 0) + c_2 F(K_1; x - e^2; 0) + c_3 F(K_1; x - e^3; 0)$$

or

$$(c_0 + c_1 z_1) F(K_1; x; z_1) + c_2 F(K_1; x - e^2; z_1) + c_3 F(K_1; x - e^3; z_1) =$$
$$= G(K_1; x; z_1) + c_0 f(x) + c_2 f(x - e^2) + c_3 f(x - e^3).$$

For $p = 2$ we get that cone $K_2 = \langle e^1, e^2 \rangle$ and for $x \in L_2 + e^3$

$$(1-\pi_1)(1-\pi_2)(c_0 + c_1 z_1 + c_2 z_2 + c_3 \delta_3) F(K_2; x; z) =$$
$$=(1-\pi_1)(1-\pi_2) G(K_2; x; z),$$

then

$$(c_0 + c_1 z_1 + c_2 z_2) F(K_2; x; z) + c_3 F(K_2; x - e^3; z) =$$
$$=(c_0 + c_2 z_2) F(K_2; x; \pi_1 z) + (c_0 + c_1 z_1) F(K_2; x; \pi_2 z) - c_0 F(K_2; x; 0)$$
$$+c_3 F(K_2; x - e^3; \pi_1 z) + c_3 F(K_2; x - e^3; \pi_2 z) - c_3 F(K_2; x - e^3; 0) +$$
$$+G(K_2; x; z),$$

For $p = 3$ we get that cone $K_3 = \langle e^1, e^2, e^3 \rangle$ and for $x \in L_3 = \varnothing$

$$
\begin{aligned}
&(1-\pi_1)(1-\pi_2)(1-\pi_3)(c_0 + c_1 z_1 + c_2 z_2 + c_3 z_3) F(K_3; x; z) = \\
=&(1-\pi_1)(1-\pi_2)(1-\pi_3) G(K_3; x; z),
\end{aligned}
$$

then

$$
\begin{aligned}
&(c_0 + c_1 z_1 + c_2 z_2 + c_3 z_3) F(z_1, z_2, z_3) - (c_0 + c_2 z_2 + c_3 z_3) F(0, z_2, z_3) - \\
&-(c_0 + c_1 z_1 + c_3 z_3) F(z_1, 0, z_3) - (c_0 + c_1 z_1 + c_2 z_2) F(z_1, z_2, 0) + \\
&+(c_0 + c_1 z_1) F(z_1, 0, 0) + (c_0 + c_2 z_2) F(0, z_2, 0) + (c_0 + c_3 z_3) F(0, 0, z_3) - \\
&-c_0 F(0, 0, 0) = 0.
\end{aligned}
$$

Fairmont State University

West Virginia's Fairmont State University welcomes first Fulbright Scholar in more than two decades

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FAIRMONT, W.Va. (WV News) — Earlier this semester, Fairmont State University welcomed its first Fulbright Scholar onto campus since 1999.

Fairmont State University College of Science & Technology faculty member Tom Cuchta and Fulbright Scholar Alexander Lyapin.

Courtesy of Fairmont State University