Some open and elementary problems in number theory

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Abstract

I will give a report and a list of open and elementary problems in number theory. Some are well-known, still open, and have been investigated by many authors. Some may be less well known and some are from my own list of problems. I would like to take this opportunity to share with the audiences the problems that might interest them. Some may be difficult but there are a few which are easy enough to assign to a student or obtain a complete answer within one semester. You can choose to work on any of this problem and publish the results all by yourself, with your colleagues, or with your students. I would only ask that after the formal publication of those results, please send me the news so that I can easily update my data. Thanks in advance!

Problem 1: Analogue of primes in arithmetic progressions and arithmetic progressions of primes

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- (Euclid) There are infinitely many primes
  (Euler) $\sum_p \frac{1}{p}$ diverges.
  (Dirichlet) If $(a, q) = 1$, then

$$\sum_{p \equiv a \pmod q} \frac{1}{p} \quad \text{diverges.}$$

So there are infinitely many primes in the arithmetic progression $\{a + qn \mid n \in \mathbb{N}\}$ whenever $(a, q) = 1$.

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("Dual"of Dirichlet's theorem: Green-Tao theorem)

Primes contains arbitrarily long arithmetic progressions. If k is given, then there is an arithmetic progression of primes of length k.

Question

- Q3 Can we find a constructive proof of Green-Tao theorem? Can we find an algorithm for Green-Tao theorem if k is given and is not too large?

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For example, $\displaystyle\sum_{\substack{p \le x \\ p \equiv a \pmod{q}}} 1 \sim \frac{x}{\varphi(q)\log x}$ if $(a, q) = 1$.

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- Q5 Does (a_n) contains arbitrarily long arithmetic progression? If not, can we find the length of the longest one.

In the case of the Fibonacci sequence, Q5 is known: $(0, 1, 2, 3)$ is the longest arithmetic progression. But Q4 is known only when q is given explicitly or when q is a power of $2, 3$ or 5, or something similar to these.

You can pick any sequence you like or find what you may like in OEIS.

Consider the Fibonacci sequence

- Burr (1971) Yes for $q = 5^k$, $2 \cdot 5^k$, $4 \cdot 5^k$, $3^3 \cdot 5^k$, $6 \cdot 5^k$, $7 \cdot 5^k$, $14 \cdot 5^k$.

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- R. Bundschuh and P. Bundschuh (2011) yes and no for $q = 3^k$ (the answer depends on a).

Problem 2: Generalized perfect numbers

Let $\sigma(n)$ be the sum of positive divisors of n. Then n is called perfect if $\sigma(n) = 2n$.

Example 6, 28, 496, 8128, 33550336, ... (A000396 in OEIS)

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Question

- Q6 Are there infinitely many even perfect numbers? Are there infinitely many Mersenne primes?
- Q7 Is there any odd perfect number?

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Question

- Q8 Can we classify all even perfect-numbers? or extend the above results?

Another generalization

n is a multiply perfect number if $n \mid \sigma(n)$.

n is a generalized multiply perfect number if $n \mid \sigma_k(n)$ for some k.

- Jiang (2018) Let $n = 2^a p^b$ where $a, b > 1$ and p is odd. Then
$$n \mid \sigma_3(n) \Leftrightarrow n \text{ is an even perfect number and } n \neq 28.$$

Question

- Q9 What are the corresponding results for $n \mid \sigma_4(n)$, $n \mid \sigma_5(n)$, $n \mid \sigma_6(n)$, ..., or $n \mid \sigma_k(n)$ for a given k?
- Q10 Can we classify n such that $n \mid \sigma_3(n)$ with $\omega(n) = 2$, $\omega(n) = 3$, ..., or $\omega(n) = k$ for a given k? (May be characterize is a more suitable word than classify).

Problem 3: The order of appearance

Let P and Q be integers. The fundamental Lucas sequence $U = U(P, Q)$ with parameters P and Q is defined by the recurrence relation $U_n = PU_{n-1} - QU_{n-2}$ for $n \geq 2$ with the initial values $U_0 = 0$ and $U_1 = 1$. If $U_2U_3U_4U_6 \neq 0$, then the sequence U is said to be nondegenerate.

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Example

If $P = 1$ and $Q = -1$, then U is the Fibonacci sequence.
If $P = 2$ and $Q = -1$, then U is the sequence of Pell numbers.
If $P = 6$ and $Q = 1$, then U is the sequence of balancing numbers.

The order of appearance

Let $U = U(P, Q)$ be a fundamental Lucas sequence and n a positive integer. The order (or rank) of appearance of n in U, denote by $\rho(n)$, is the smallest positive integer k such that $n \mid U_k$. In the case U is the Fibonacci sequence, we write $z(n)$ for $\rho(n)$.

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Example

n	0	1	2	3	4	5	6	7	8	9	10	11	12
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144

$$z(1) = 1, z(2) = 3, z(3) = 4, z(4) = 6, z(5) = 5,$$

$$z(6) = 12, z(7) = 8, z(8) = 6, \text{ and } z(9) = 12.$$

(Well-known) $z(F_n) = n$ and $z(L_n) = 2n$ for all $n \geq 3$.

There is no general formula for $z(m)$ for all $m \geq 1$. But some special cases have been obtained.

$$z(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) = [z(p_1^{a_1}), z(p_2^{a_2}), \ldots, z(p_k^{a_k})].$$

(But there is no formula for $z(p^a)$ for all a and p.)

D. Marques obtains $z(m)$ when

$$m = F_n^k, \quad L_n^k, \quad F_nF_{n+1}, \quad F_nF_{n+1}F_{n+2}, \quad F_nF_{n+1}F_{n+2}F_{n+3},$$
$$L_nL_{n+1}, \quad L_nL_{n+1}L_{n+2}, \quad L_nL_{n+1}L_{n+2}L_{n+3}, \quad etc$$

D. Marques

(i) For $n \geq 3$,

$$z(F_n F_{n+1}) = n(n+1).$$

(ii) For $n \geq 2$,

$$z(F_n F_{n+1} F_{n+2}) = \begin{cases} n(n+1)(n+2), & \text{if } n \equiv 1 \pmod 2, \\ \frac{n(n+1)(n+2)}{2}, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

(iii) For $n \geq 1$,

$$z(F_n \cdots F_{n+3}) = \begin{cases} \frac{n(n+1)(n+2)(n+3)}{2}, & \text{if } n \not\equiv 0 (\bmod 3), \\ \frac{n(n+1)(n+2)(n+3)}{3}, & \text{if } n \equiv 0, 9 (\bmod 12), \\ \frac{n(n+1)(n+2)(n+3)}{6}, & \text{if } n \equiv 3, 6 (\bmod 12). \end{cases}$$

What is $z(m)$ when $m = F_n F_{n+1} \cdots F_{n+k}$ and $k \geq 4$?

N. Khaochim and P

Let $n \geq 1$ and $b = F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}$. Then

$$z(b) = \begin{cases} \frac{n(n+1)(n+2)(n+3)(n+4)}{2}, & \text{if } n \equiv 1, 7 \pmod{12}; \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{3}, & \text{if } n \equiv 9, 11 \pmod{12}; \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{4}, & \text{if } n \equiv 10 \pmod{12} \\ & \text{ or } n \equiv 0, 20, 48, 68 \pmod{72}; \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{6}, & \text{if } n \equiv 3, 5 \pmod{12}; \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{8}, & \text{if } n \equiv 4 \pmod{12} \\ & \text{ or } n \equiv 12, 32, 36, 56 \pmod{72}; \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{12}, & \text{if } n \equiv 2, 6 \pmod{12} \\ & \text{ or } n \equiv 24, 44 \pmod{72}; \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{24}, & \text{if } n \equiv 8, 60 \pmod{72}. \end{cases}$$

Let $n \geq 1$, $a = [n, n+1, n+2, n+3, n+4]$, and $b = F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}$. Then

$$z(b) = \begin{cases} a, & \text{if } n \equiv 1, 2, 3, 4, 5, 6, 7, 10 \pmod{12}, \\ & \text{ or } n \equiv 8, 60 \pmod{72}; \\ 2a, & \text{if } n \equiv 9, 11 \pmod{12}, \\ & \text{ or } n \equiv 24, 44 \pmod{72}; \\ 3a, & \text{if } n \equiv 12, 32, 36, 56 \pmod{72}; \\ 6a, & \text{if } n \equiv 0, 20, 48, 68 \pmod{72}. \end{cases}$$

Let $n \geq 1$, $a = [n, n+1, n+2, n+3, n+4]$, and $b = F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}$. Then

$$z(b) = \begin{cases} a, & \text{if } n \equiv 1 \pmod 3 \\ & \quad \text{or } n \equiv 2, 3, 5, 6 \pmod{12}; \\ 2a, & \text{if } n \equiv 9, 11 \pmod{12}; \\ \frac{72a}{(8,n)(9,n+1)}, & \text{if } n \equiv 8 \pmod{12}; \\ \frac{72a}{(8,n+4)(9,n+3)}, & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Let $n \geq 1$, $a = [n, n+1, \ldots, n+5]$, $b = F_n F_{n+1} \cdots F_{n+5}$, and $c = (5, n)$. Then

$$z(b) = \begin{cases} ac, & \text{if } n \equiv 1,2,3,4,5,6 \pmod{12} \\ 2ac, & \text{if } n \equiv 9, 10 \pmod{12}; \\ \frac{72(5,n)a}{(8,n+|r-8|)(9,n+|r-9|)}, & \text{if } n \equiv r \pmod{12} \\ & \quad \text{and } r \in \{7, 8, 12\}; \\ \frac{72(5,n)a}{(8,n+5)(9,n+4)}, & \text{if } n \equiv 11 \pmod{12}. \end{cases}$$

Let $n \geq 1$, $a = [n, n+1, \ldots, n+6]$, $b = F_n F_{n+1} \cdots F_{n+6}$, and $c = (5, n(n+1))$. Then $z(b) =$

$$\begin{cases} ac, & \text{if } n \equiv 1,2,3,4,5 \pmod{12}; \\ \frac{(64)(27)ac}{(64,n+2)(27,n(n+3))}, & \text{if } n \equiv 6 \pmod{24}; \\ \frac{(8)(27)ac}{(27,n(n+3))}, & \text{if } n \equiv 18 \pmod{24}; \\ \frac{72ac}{(8,n-r)(9,n-r)}, & \text{if } n \equiv r \pmod{12} \text{ and } r \in \{7,8\}; \\ 4ac, & \text{if } n \equiv 9 \pmod{12}; \\ \frac{72ac}{(8,n+6)(9,n+5)}, & \text{if } n \equiv 10 \pmod{12}; \\ \frac{72ac}{(8,n+5)(9,n+4)}, & \text{if } n \equiv 11 \pmod{12}; \\ \frac{(64)(27)ac}{(64,n+4)(27,(n+3)(n+6))}, & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Open problems

$z(m) = ?$ when

$$m = F_n L_{n+1} F_{n+2} \cdots F_{n+k}$$

$$m = F_n L_{n+1} F_{n+2} \cdots L_{n+k}$$

$$m = L_n F_{n+1} L_{n+2} \cdots L_{n+k}$$

$$m = \binom{n}{k}$$

$\vdots$

The first 3 cases are easy and the algorithm given by Khaochim and P still works. For $m = \binom{n}{k}$, may be some special conditions are needed. For example, if $k = 1$, three is nothing we can do. May be, central binomial coefficients are more interesting.

Question

- Q11 Can we extend these results to any fundamental Lucas sequence and to its companion?

Example

Patel, Panda, etc are trying to do this for balancing and Lucas balancing numbers.

Problem 4: (From) Exact divisibility by powers of the Fibonacci and Lucas numbers (to...)

Definition

The sequence F_n of Fibonacci numbers is defined by the recurrence relation: $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ and $F_1 = 1$, $F_2 = 1$.
The sequence L_n of Lucas numbers is defined by the recurrence relation: $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$ and $L_1 = 1$, $L_2 = 3$.

Example

$$\begin{bmatrix} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ F_n & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 \\ L_n & 1 & 3 & 4 & 7 & 11 & 18 & 29 & 47 & 76 \end{bmatrix}$$

Well-known

$$m \mid n \Rightarrow F_m \mid F_n,$$
$$F_m \mid F_n \text{ and } m \neq 2 \Rightarrow m \mid n,$$
$$\gcd(F_m, F_n) = F_{\gcd(m,n)}.$$

A divisibility result which is used in Matiyasevich's solution to Hilbert's 10th Problem (1970)

$$F_n^2 \mid F_{nm} \Leftrightarrow F_n \mid m \tag{1}$$

Hoggatt and Bicknell-Johnson (1977) gives another proof of (1) and extend it to higher powers.

If n is odd, then $F_m^3 \mid F_{mn} \Leftrightarrow F_m^2 \mid n$.
If $3 \nmid m$, then $F_m^3 \mid F_{mn} \Leftrightarrow F_m^2 \mid n$.
If $n \equiv \pm 1 \pmod 6$ and $s = 2, 3, 4, 5, 6$, then

$$F_m^s \mid F_{mn} \Leftrightarrow F_m^{s-1} \mid n.$$

If $3 \nmid m$, $4 \nmid m$, and $s = 2, 3, 4, 5, 6$, then

$$F_m^s \mid F_{mn} \Leftrightarrow F_m^{s-1} \mid n.$$

The result is not complete.

- Tangboonduangjit and Wiboonton,
Panraksa, Tangboonduangjit and Wiboonton, consider a subsequence $(G_k(n))_{k\geq 1}$ of the Fibonacci sequence $G_1(n) = F(n)$, $G_k(n) = F(nG_{k-1}(n))$

$$F(n), F(nF(n)), F(nF(nF(n))), \ldots$$

$$\begin{aligned} F_n | G_1(n) &\Rightarrow F_n^2 \mid F(nG_1(n)) = G_2(n) \\ &\Rightarrow F_n^3 \mid F(nG_2(n)) = G_3(n) \\ &\Rightarrow F_n^4 \mid F(nG_3(n)) = G_4(n) \end{aligned}$$

$F_n^k \mid G_k(n)$ for all $k \geq 1$. (Tangboonduangjit and Wiboonton (2012))

$F_n^k \parallel G_k(n)$ for all $k \geq 1$. (Panraksa, Tangboonduangjit and Wiboonton (2012))

$$\frac{G_2(n)}{F_n^2} \equiv \begin{cases} \frac{F_{n-3}}{2}, & \text{if } 3 \mid n; \\ 1, & \text{if } 3 \nmid n. \end{cases}$$

$$\frac{G_3(n)}{F_n^3} \equiv \begin{cases} 1, & \text{if } 3 \nmid n \text{ and } 4 \nmid n; \\ F_{n-1}, & \text{if } 3 \nmid n \text{ and } 4 \mid n; \\ \frac{(-1)^n}{4} F_{n-3}^2, & \text{if } 3 \mid n. \end{cases}$$

Onphaeng and P (2014)

$G(1,n,m) = F_n^m$, $G(k+1,n,m) = F(nG(k,n,m))$.

$$F_n^m, F_{nF_n^m}, F_{nF_{nF_n^m}}, F_{nF_{nF_{nF_n^m}}}, \dots$$

$$F_n^{k+m-1} \mid G(k,n,m) \text{ for every} k,n,m \geq 1$$

$$F_n^{k+m-1} \parallel G(k,n,m) \text{ for } k \geq 2, n \geq 4, m \geq 1$$

$$\frac{G(k,n,m)}{F_n^{k+m-1}} \equiv \begin{cases} 1, & \text{if } 2 \mid k \text{ and } 3 \nmid n \text{ or if } 2 \nmid k,\ 3 \nmid n \text{ and} \\ F_{n-1}, & \text{if } 2 \nmid k,\ 3 \nmid n \text{ and } 4 \mid n; \\ \left(\frac{F_{n-3}}{2}\right)^{k-1}, & \text{if } 2 \nmid k,\ 3 \mid n \text{ and } m \geq 2 \text{ or} \\ & \text{if } 2 \mid k,\ 3 \mid n \text{ and } m = 1; \\ (-1)^n \left(\frac{F_{n-3}}{2}\right)^{k-1}, & \text{if } 2 \mid k,\ 3 \mid n \text{ and } m \geq 2 \text{ or} \\ & \text{if } 2 \nmid k,\ 3 \mid n \text{ and } m = 1. \end{cases}$$

P (2014)

For $n \geq 3$, we have

(i) if $F_n^k \parallel m$ and $n \not\equiv 3 \pmod 6$, then $F_n^{k+1} \parallel F_{nm}$;

(ii) if $F_n^k \parallel m$, $n \equiv 3 \pmod 6$ and $\frac{F_n^{k+1}}{2} \nmid m$, then $F_n^{k+1} \parallel F_{nm}$;

(iii) if $F_n^k \parallel m$, $n \equiv 3 \pmod 6$ and $\frac{F_n^{k+1}}{2} \mid m$, then $F_n^{k+2} \parallel F_{nm}$.

P (2014)

Let m be an odd integer. Then

(i) if $L_n^k \mid m$, then $L_n^{k+1} \mid L_{nm}$;

(ii) if $n \geq 2$ and $L_n^k \parallel m$, then $L_n^{k+1} \parallel L_{nm}$.

P (2014)

Let m be even and $n \geq 2$. Then the following statements hold.

(i) If $L_n^k \mid m$, then $L_n^{k+1} \mid F_{nm}$.

(ii) If $L_n^k \parallel m$ and $n \not\equiv 0 \pmod 3$, then $L_n^{k+1} \parallel F_{nm}$.

(iii) If $L_n^k \parallel m$, $n \equiv 0 \pmod 6$ and $\frac{L_n^{k+1}}{2} \nmid m$, then $L_n^{k+1} \parallel F_{nm}$.

(iv) If $L_n^k \parallel m$, $n \equiv 0 \pmod 6$ and $\frac{L_n^{k+1}}{2} \mid m$, then $L_n^{k+2} \mid F_{nm}$.

(v) If $L_n^k \parallel m$, $n \equiv 3 \pmod 6$ and $\frac{L_n^{k+1}}{4} \nmid m$, then $L_n^{k+1} \parallel F_{nm}$.

(vi) If $L_n^k \parallel m$, $n \equiv 3 \pmod 6$ and $\frac{L_n^{k+1}}{4} \mid m$, then $L_n^{k+2} \mid 4F_{nm}$.

Onphaeng and P

Let k, m, n be positive integers and $n \geq 3$. Then the following statements hold.

(i) If $F_n^{k+1} \mid F_{nm}$ and $n \not\equiv 3 \pmod 6$, then $F_n^k \mid m$.

(ii) If $F_n^{k+1} \mid F_{nm}$ and $n \equiv 3 \pmod 6$, then $F_n^k \mid 2m$ and $F_n^{k-1} \mid m$.

(iii) If $F_n^{k+1} \mid F_{nm}$, $n \equiv 3 \pmod 6$, and $2^k | m$, then $F_n^k \mid m$.

Onphaeng and P

Let k, m, n be positive integers and $n \geq 3$. Then the following statements hold.

(i) If $F_n^{k+1} \parallel F_{nm}$ and $n \not\equiv 3 \pmod 6$, then $F_n^k \parallel m$.

(ii) If $F_n^{k+1} \parallel F_{nm}$, $n \equiv 3 \pmod 6$, and $2^k | \ m$, then $F_n^k \parallel m$.

(iii) If $F_n^{k+1} \parallel F_{nm}$, $n \equiv 3 \pmod 6$, and $2^k \nmid m$, then $F_n^{k-1} \parallel m$.

Onphaeng and P

Let k, m, n be positive integers and $n \geq 2$. Then the following statements hold.

(i) If $L_n^{k+1} \mid L_{nm}$, then $n \not\equiv 0 \pmod 3$, m is odd, and $L_n^k \mid m$.

(ii) If $L_n^{k+1} \parallel L_{nm}$, then $L_n^k \parallel m$.

Onphaeng and P

Let k, m, n be positive integers and $n \geq 2$. If $L_n^{k+1} | F_{nm}$, then m is even. Moreover, the following statements hold.

(i) If $L_n^{k+1} \mid F_{nm}$ and $n \not\equiv 0 \pmod 6$, then $L_n^k \mid m$.

(ii) If $L_n^{k+1} \parallel F_{nm}$ and $n \not\equiv 0 \pmod 6$, then $L_n^k \parallel m$.

(iii) If $L_n^{k+1} \mid F_{nm}$ and $n \equiv 0 \pmod 6$, then $L_n^{min\{v_2(m),k\}} \mid m$.

(iv) If $L_n^{k+1} \parallel F_{nm}$ and $n \equiv 0 \pmod 6$, then $L_n^{min\{v_2(m),k\}} \parallel m$.

Let $P, Q \in \mathbb{Z}$. Define $U_n(P, Q)$ by

$$U_0 = 0, U_1 = 1, \; U_n = PU_n - QU_{n-1}, \text{ for } n \geq 3.$$

Question

- Q12 Find the corresponding results for $U_n(P, Q)$ for all P, Q or a large class of P, Q.

Update: Define $G_1(n) = U_n$ and $G_k(n) = U(nG_{k-1}(n))$ for $k \geq 2$. The results on the exact divisibility of $G_k(n)$ by powers of U_n has been recently obtained by Panraksa and Tangboonduangjit (communication after the seminar).
Nevertheless, some of the results analogous to those of Onphaeng and P for U_n are still open.

Euler's function

$$\varphi(n) = \sum_{\substack{1 \le k \le n \\ (k,n)=1}} 1$$

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$$1, 5, 7, 11 \qquad \text{So } \varphi(12) = 4$$

$$\varphi(100) = ?$$

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

$$\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right)$$

$$\varphi(100) = 100 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{5}\right) = 40$$

Lehmer's Problem

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- Q13 If $\varphi(n) \mid n - 1$, then n is prime?

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Does the converse hold?

- Q13 If $\varphi(n) \mid n - 1$, then n is prime?

This is called Lehmer's problem on Euler's function.

Carmichael's Problem

$$\forall k \geq 2 \ \exists m \in \mathbb{N} \text{ such that}$$

$$\varphi(n) = m \text{ has exactly } k \text{ solutions.}$$

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$\exists m \in \mathbb{N},\ \left|\varphi^{-1}(m)\right| = k$

Carmichael's Problem

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$\exists m \in \mathbb{N}, \ \left|\varphi^{-1}(m)\right| = 4$

$\exists m \in \mathbb{N}, \ \left|\varphi^{-1}(m)\right| = k$

Question

- Q14 $\exists m \in \mathbb{N}, \ \left|\varphi^{-1}(m)\right| = 1$?

Palindromes

A positive integer n is said to be a palindrome in base b (or b-adic palindrome) if the representation of $n = (a_k a_{k-1} \cdots a_0)_b$ in base b with $a_k \neq 0$ has the symmetric property $a_{k-i} = a_i$ for every $i = 0, 1, 2, \ldots, k$.

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Example

The following numbers are palindromes.

- 1
- 121
- $(1001)_2$
- $(1234321)_5$

Generalized Palindromes

A positive integer n is said to be a generalized palindrome in base b (or generalized b-adic palindrome) if the representation of $n = (a_k a_{k-1} \cdots a_0)_b$ in base b has the symmetric property $a_{k-i} = a_i$ for every $i = 0, 1, 2, \ldots, k$.

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Example

The following numbers are generalized palindromes.

- $(010)_2$
- $(11)_2$
- $(0110)_2$
- $(0003000)_5$

Banks (2016)

Every natural number is the sum of at most forty-nine 10-adic palindromes.

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Cilleruelo, Luca, and Baxter (2017)

For $b \geq 5$, every natural number is the sum of at most three b-adic palindromes.

Rajasekaran, Shallit, and Smith (2017)

Every natural number is the sum of at most four 2-adic palindromes.

Rajasekaran, Shallit, and Smith (2017)

Every natural number is the sum of at most three generalized 2-adic palindromes.

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THANK YOU FOR YOUR ATTENTION