# Strongly restricted permutations

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A strongly restricted permutation $\pi$ of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$ is a permutation for which the number of permissible values of $\pi(i) - i$ for each $i \in \mathbb{N}_n$ is less than a finite number which is independent of $n$. In general, the number of such permutations $A_n$ can be obtained by evaluating the permanent of a matrix whose $(i, j)$th entry is 1 if it is permissible for $\pi(i)$ to equal $j$ and 0 otherwise. However, it is more useful to obtain a recurrence relation for $A_n$. We will look at a general technique by Baltić for obtaining such a recurrence.

*Outline*

- Restricted permutations
- Permanents
- Strongly restricted permutations
- Baltić's 5-step procedure for obtaining recursion relation
- Special case: $(r+1)$-step Fibonacci numbers

*A permutation* $\pi$ *of* $\mathbb{N}_4 = \{1, 2, 3, 4\}$

$$\begin{pmatrix} i: & 1 & 2 & 3 & 4 \\ \pi(i): & 4 & 1 & 3 & 2 \\ \pi(i) - i: & 3 & -1 & 0 & -2 \end{pmatrix}$$

## *Permutations and restricted permutations*

The number of permutations of $\mathbb{N}_n = \{1, 2, \ldots, n\}$ is

$$n(n-1)\cdots 1 = n!$$

$n!_{n\geqslant 0} = 1, 1, 2, 6, 24, 120, \ldots$

A restricted permutation refers to the case when some of the moves are not allowed. E.g., a *derangement* is a permutation with no 'fixed points' – all the numbers must move. The number of derangements $D_n$ satisfies

$$D_n = n! \sum_{k=0}^{n} \frac{(-)^k}{k!}$$

or the recurrence

$$D_n = nD_{n-1} + (-1)^n, \quad D_0 = 1$$

$D_{n\geqslant 0} = 1, 0, 1, 2, 9, 44, \ldots$ (sequence A000166 in Sloane's OEIS).

*Permanent of an $n \times n$ matrix $\mathbf{M}$*

$$\det \mathbf{M} = \sum_{i_1, i_2, \ldots, i_n} \varepsilon_{i_1 i_2 \ldots i_n} M_{1 i_1} M_{2 i_2} \ldots M_{n i_n}$$

$$= \sum_{i=1}^{n!} \operatorname{sgn}(\pi_i) M_{1\pi_i(1)} M_{2\pi_i(2)} \ldots M_{n\pi_i(n)}$$

where $\varepsilon_{i_1 i_2 \ldots i_n}$ is the permutation symbol; and $\pi_i$, for $1 \leqslant i \leqslant n!$, are functions giving all the $n!$ permutations of $\mathbb{N}_n$; and $\operatorname{sgn}(\pi_i)$ is $1$ ($-1$) if $\pi_i$ is an even (odd) permutation.

The permanent of a matrix is like the determinant but with all plus signs:

$$\operatorname{per} \mathbf{M} = \sum_{i=1}^{n!} M_{1\pi_i(1)} M_{2\pi_i(2)} \ldots M_{n\pi_i(n)}$$

As with determinants, evaluate $\operatorname{per} \mathbf{M}$ by expanding along a row or column.

*Evaluating the number of restricted permutations using permanents*

$M_{ij} = 1_{i \text{ is allowed to move to } j \text{ in a permutation}}$. Then the number of restricted permutations is $\operatorname{per}\mathbf{M}$.

E.g., if all moves are allowed (no restrictions) then $M_{ij} = 1$ and $\operatorname{per}\mathbf{M} = n!$.

E.g., derangements of $\{1,2,3\}$ and $\{1,2,3,4\}$:

$$\operatorname{per}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 2, \qquad \operatorname{per}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = 9$$

## *Strongly restricted permutations*

Lehmer 1970: A *strongly restricted permutation* $\pi$ of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$ is a permutation for which the number of permissible values of $\pi(i) - i$ for each $i \in \mathbb{N}_n$ is less than a finite number which is independent of $n$.

E.g., $\pi(i) - i \in \{-1, 0, 1, 2\}$. The number of such strongly restricted permutations for $n = 3$ and $n = 4$ is then

$$\mathrm{per}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 2! + 2! = 4, \qquad \mathrm{per}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = 4 + 2! + 1 = 7$$

[Lehmer also defined *weakly restricted permutations* as those for which $n-$(number of permissible values of $\pi(i) - i$) is less than a finite number independent of $n$. Derangements are an example.]

## *Procedure for finding recursion relation*

Baltić 2010: developed a 5-step procedure for finding the recursion relation for $A_n$, the number of strongly restricted permutations of $\mathbb{N}_n$ given the set $\mathcal{W}$ of allowed values of $\pi(i) - i$.

Let $-k = \min(\pi(i) - i)$, $r = \max(\pi(i) - i)$.

WLOG, we can assume that $r \geqslant k$. Also, we must have $k > 0$.

We first look at the case when $\mathcal{W}$ has no gaps:

$$\pi(i) - i \in \mathcal{W} = \{-k, -k+1, \ldots, r-1, r\}$$

$$\pi(i) - i \in \mathcal{W} = \{-k, -k+1, \ldots, r-1, r\}$$

STEP 1. Create $\mathcal{C}$, the set of all $(k+1)$-element combinations of the set $\mathbb{N}_{r+k+1}$ that contain the element $k+r+1$. [$|\mathcal{C}| = \binom{r+k}{k}$.] E.g., if $k = r = 2$,

$$\mathcal{C} = \{345, 245, 235, 145, 135, 125\}$$

where, e.g., 345 denotes the combination $\{3, 4, 5\}$.

STEP 2. Introduce an integer sequence $a_C(n)$ for each combination $C \in \mathcal{C}$. E.g., $a_{345}(n)$, $a_{245}(n)$, etc.

The number $A_n$ of strongly restricted permutations of $\mathbb{N}_n$ will end up being $a_{\{r+1,\ldots,r+k+1\}}(n)$ (i.e., $A_n = a_{345}(n)$ in the current example).

STEP 3. Apply following mapping to each combination $C \in \mathcal{C}$:

$$\phi(C) = \begin{cases} \phi_1(C), & 1 \in C, \\ \phi_2(C), & 1 \notin C, \end{cases}$$

where

$$\phi_1(\{1, c_2, \ldots, c_k, c_{k+1}\}) = \{\{c_2 - 1, \ldots, c_k - 1, c_{k+1} - 1, r + k + 1\}\},$$

e.g., $\phi(145) = 345$, $\phi(135) = 245$, $\phi(125) = 145$, and

$$\phi_2(\{c_1, c_2, \ldots, c_k, c_{k+1}\}) = \{C_1, C_2, \ldots, C_k, C_{k+1}\},$$
$$C_i = \{c_1 - 1, \ldots, c_{i-1} - 1, c_{i+1} - 1, \ldots, c_{k+1} - 1, r + k + 1\}.$$

E.g., $\phi(345) = \{345, 245, 235\}$, $\phi(245) = \{345, 145, 135\}$, $\phi(235) = \{245, 145, 125\}$.

STEP 4. Create a system of $\binom{k+r}{k}$ recurrence relations:

$$a_C(n) = \sum_{C' \in \phi(C)} a_{C'}(n-1)$$

with initial conditions $a_C(0) = 1_{C=\{r+1,\ldots,r+k+1\}}$.

E.g.,

$$a_{345}(n+1) = a_{345}(n) + a_{245}(n) + a_{235}(n),$$

$$a_{245}(n+1) = a_{345}(n) + a_{145}(n) + a_{135}(n),$$

$$a_{235}(n+1) = a_{245}(n) + a_{145}(n) + a_{125}(n),$$

$$a_{145}(n+1) = a_{345}(n), \quad a_{135}(n+1) = a_{245}(n), \quad a_{125}(n+1) = a_{145}(n),$$

with $a_{345}(0) = 1$, $a_{245}(0) = \cdots = a_{125}(0) = 0$.

## *Aside: generating functions*

The pre-reduction generating function $g(z)$ for the recursion relation

$$A_n = \delta_{n,0} + \sum_{m>0} (\alpha_m \delta_{n,m} + \beta_m A_{n-m})$$

where $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise, and $A_{n<0} = 0$ is given by

$$g(z) = \frac{1 + \sum_{m>0} \alpha_m z^m}{1 - \sum_{m>0} \beta_m z^m}.$$

STEP 5. Solve the system of recurrence relations to find $a_{\{r+1,\dots,r+k+1\}}(n)$ which is $A_n$, the number of strongly restricted permutations.

This is easiest to do via generating functions. Let $g_C(z)$ be the generating function corresponding to the recurrence relation for $a_C(n)$ and its boundary condition $a_C(0)$. In our example, this gives

$$\begin{aligned} g_{345}(z) - 1 &= z(g_{345}(z) + g_{245}(z) + g_{235}(z)), \\ g_{245}(z) &= z(g_{345}(z) + g_{145}(z) + g_{135}(z)), \\ g_{235}(z) &= z(g_{245}(z) + g_{145}(z) + g_{125}(z)), \\ g_{145}(z) &= zg_{345}(z), \quad g_{135}(z) = zg_{245}(z), \quad g_{125}(z) = zg_{145}(z), \end{aligned}$$

from which the required generating function is

$$g_{345}(z) = \frac{1-z}{1-2z-2z^3+z^5}$$

Hence $A_n = 2A_{n-1} + 2A_{n-3} - A_{n-5} + \delta_{n,0} - \delta_{n,1}$, with $A_{n<0} = 0$.

## *Proof that* $A_n = a_{\{r+1,\ldots,r+k+1\}}(n)$

Let $\mathcal{M}$ be the set of $n \times n$ (0,1)-matrices $\mathbf{M}$ which have the following properties:

- $M_{ij} = 1_{j \leqslant d_i}$ for $i = 1, \ldots, k+1$ where $d_i \geqslant 1$ strictly increase with $i$ and $d_{k+1} = r + k + 1$;
- $M_{ij} = 1_{-k \leqslant j-i \leqslant r}$ for $i = k+2, \ldots, n$.

There is a 1-1 correspondence between elements of $\mathcal{C}$ and $\mathcal{M}$ for $n > r$: the combination $C$ corresponding to $\mathbf{M}$ is $\{d_1, \ldots, d_{k+1}\}$. We can therefore label the matrices using $C$; we write $\mathbf{M}_C$.

It can be seen that

$$A_n = \operatorname{per} \mathbf{M}_{\{r+1,\ldots,r+k+1\}}$$

For example, if $n = 8$ in the $k = r = 2$ case (i.e., $\pi(i) - i \in \{-2, 1, 0, 1, 2\}$)

$$\mathbf{M}_{345} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

In general, $a_C(n) = \operatorname{per} \mathbf{M}_C$ since the procedure for obtaining the recurrence relations corresponds to expanding the permanent about the first row (which gives $\phi_1$) or the first column (which gives $\phi_2$).

## *Finding* $A_n$ *when* $-k \leqslant \pi(i) - i \leqslant r$ *and* $\pi(i) - i \notin I$

$I$ is a fixed subset of $\{-k+1, -k+2, \ldots, r-1\}$. E.g., $k = 2$, $r = 3$, $I = \{-1, 2\}$, i.e., $\pi(i) - i \in \{-2, 0, 1, 3\}$.

Find $\mathcal{C}$ as before. Also form the set $P = \{r + 1 - i \mid i \in I\}$. E.g.,

$$\mathcal{C} = \{456, 356, 346, 256, 246, 236, 156, 146, 136, 126\} \quad P = \{2, 5\}$$

For $C \ni 1$, the mapping $\phi(C)$ stays the same, i.e., $\phi(C) = \phi_1(C)$. E.g., $\phi(156) = 456,\ \ \phi(146) = 356,\ \ \phi(136) = 256,\ \ \phi(126) = 156.$

For $C \not\ni 1$, we have $\phi(C) = \phi_2^m(C)$ where $m$ is the number of elements in $C$ that are also in $P$. Hence $m = 0, \ldots, |I|$. $\phi_2^m$ is the same as $\phi_2$ except any combination $C_i$ such that $c_i \in P$ is removed. [Thus $\phi_2^0 = \phi_2$.] E.g., $\phi(346) = \{356, 256, 236\},\ \ \phi(456) = \{456, 346\},\ \ \phi(356) = \{456, 246\},$ $\phi(246) = \{156, 136\},\ \ \phi(236) = \{156, 126\},\ \ \phi(256) = 146.$

As before, form recursion relations via

$$a_C(n+1) = \sum_{C' \in \phi(C)} a_{C'}(n)$$

with $a_C(0) = 1_{C=\{r+1,\ldots,r+k+1\}}$. Then $A_n = a_{C=\{r+1,\ldots,r+k+1\}}(n)$. E.g.,

$$a_{456}(n+1) = a_{456}(n) + a_{346}(n), \quad a_{356}(n+1) = a_{456}(n) + a_{246}(n), \quad \cdots$$

Combine and obtain recursion relation using generating functions. E.g.,

$$g_{456}(z) = \frac{1 - z^5}{1 - z - z^3 - z^4 - 4z^5 + z^6 - z^7 + z^9 + z^{10}}$$

Hence

$$A_n = A_{n-1} + A_{n-3} + A_{n-4} + 4A_{n-5} - A_{n-6} + A_{n-7} - A_{n-9} - A_{n-10} + \delta_{n,0} - \delta_{n,5}$$

with $A_{n<0} = 0$. $A_{n \geqslant 0} = 1, 1, 1, 2, 4, 9, 15, 25, 46, 84, 156, \ldots$ is A080004.

*Special case:* $\pi(i) - i \in \{-1, 0, 1, \ldots, r\}$

Then $k = 1$ and $\mathcal{C} = \{\{1, r+2\}, \{2, r+2\}, \ldots, \{r+1, r+2\}\}$. Then

$$\phi(\{1, r{+}2\}) = \{r{+}1, r{+}2\}, \quad \phi(\{i, r{+}2\}) = \{\{i{-}1, r{+}2\}, \{r{+}1, r{+}2\}\}$$

for $i = 2, \ldots, r+1$. Hence, writing $A_n = a_{\{r+1,r+2\}}$,

$$A_n = A_{n-1} + a_{\{r,r+2\}}(n-1),\ a_{\{r,r+2\}}(n) = A_{n-1} + a_{\{r-1,r+2\}}(n-1)$$
$$a_{\{r-1,r+2\}}(n) = A_{n-1} + a_{\{r-2,r+2\}}(n-1),\ \cdots\ a_{\{1,r+2\}}(n) = A_{n-1}$$

Thus we get the $(r+1)$-step Fibonacci sequence:

$$A_n = A_{n-1} + A_{n-2} + \cdots + A_{n-(r+1)} + \delta_{n,0}, \quad A_{n<0} = 0.$$

E.g., for $\pi(i) - i \in \{-1, 0, 1, 2\}$ we get the Tribonacci sequence $A_{n\geqslant 0} = 1, 1, 2, 4, 7, \ldots$.

## *References*


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