Riordan arrays

16:00 Wednesday 8th May 2019, Room 1210, MUIC

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A $(p(x), q(x))$ Riordan array, where $p(x) = p_0 + p_1 x + p_2 x^2 + \dots$, $q(x) = q_1 x + q_2 x^2 + \dots$, is an infinite lower triangular matrix whose $(n \geqslant 0, k \geqslant 0)$th entry is the coefficient of x^n in $p(x)(q(x))^k$. We will discuss basic properties of Riordan arrays, including their group structure, along with some applications that include obtaining identities and analysing lattice paths.

Acknowledgements

Ken Edwards

- Riordan arrays seminar on youtube by M. Zeleke
- Sprugnoli R (1994) "Riordan arrays and combinatorial sums". *Discrete Mathematics*, **132**, 267-90.
- Shapiro L (2005) "A survey of the Riordan group".

Riordan arrays: outline

- Definition and simple properties
- Pascal's triangle as an example
- History
- The Riordan group
- Partial sum theorem
- Generation of identities
- Riordan arrays via A- and Z-sequences
- Fundamental theorem of Riordan arrays
- Applications to lattice paths

Definition *(as used in the OEIS and recent literature)* *and simple properties*

A $(p(x), q(x))$ Riordan array, where $p(x) = p_0 + p_1 x + p_2 x^2 + \ldots,$ $q(x) = q_1 x + q_2 x^2 + \ldots,$ is an infinite lower triangular matrix, which we'll denote by (p, q), whose $(n \geqslant 0, k \geqslant 0)$th entry $(p, q)_{n,k} = [x^n]p(x)q(x)^k$.

$[x^n] \ldots$ means the coefficient of x^n in the power series of ....

- the first term in the series for pq^k is $p_0 q_1^k x^k$. Hence
  - above leading diagonal: $(p, q)_{n,k>n} = 0$
  - leading diagonal: $(p, q)_{n,n} = p_0 q_1^n$
- first ($k = 0$) column: $(p, q)_{n,0} = p_n$
- kth column gives the coefficients of the power series for pq^k

As Riordan arrays have applications in combinatorics, p and q tend to have integer coefficients and can be thought of as generating functions of integer sequences: the nth term in the sequence $\{p_n\}_{n \geqslant 0}$ is $[x^n]p(x) = p_n$.

Example: Pascal's triangle

The $(1/(1-x), x/(1-x))$ Riordan array is Pascal's triangle,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix},$$

which has $\binom{n}{k}$ as its (n, k)th entry.

Example: Proof that the $(1/(1-x), x/(1-x))$ Riordan array is Pascal's triangle

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right)_{n,k} = [x^n]\frac{1}{1-x}\frac{x^k}{(1-x)^k} = [x^n]x^k(1-x)^{-(k+1)}$$

$$= [x^{n-k}]\Bigg(1 + (k+1)x + \frac{(k+1)(k+2)}{2!}x^2 + \cdots$$

$$+ \frac{(k+1)(k+2)\cdots(k+n-k)}{(n-k)!}x^{n-k} + \cdots\Bigg)$$

$$= \frac{k!(k+1)(k+2)\cdots(n)}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

History

- Introduced by Lou Shapiro et al in 1989 who named them in honour of John Riordan (1903–88) in recognition of his contributions to combinatorics.
- Riordan arrays are generalizations of ‘renewal arrays’ introduced by Rogers in 1978 as generalizations of Pascal’s triangle and similar triangles.
- Sprugnoli pointed out the connection between Riordan arrays and combinatorial sums in 1994.
- Donatella Merlini
- Paul Barry

The Riordan group

The Riordan group is the set of $(p = p_0 + p_1x + \ldots, q = q_1x + \ldots)$ Riordan arrays such that $p_0 = 1$ and $q_1 \neq 0$ with the group operation on two elements (p, q) and (P, Q) given by

$$(p(x), q(x))(P(x), Q(x)) = (p(x)P(q(x)), Q(q(x)))$$

which corresponds to matrix multiplication of the Riordan arrays

identity element $(1, x)$ (for which $(1, x)_{n,k} = [x^n]x^k = \delta_{n,k}$, the identity matrix)

inverse $(p(x), q(x))^{-1} = (1/p(q^{-1}(x)), q^{-1}(x))$ (inverse of matrix)

closed since $[x^0]p(x)P(q(x)) = 1$ and $[x^0]Q(q(x)) = 0$

associative

Example of inverse: ‘inverse’ of Pascal’s triangle

For the Pascal’s triangle Riordan array, $p(x) = 1/(1-x)$, $q(x) = x/(1-x)$ and so $x = q(x)/(1+q(x))$ and $q^{-1}(x) = x/(1+x)$. So $(1/(1-x), x/(1-x))^{-1} = (1-q^{-1}(x), q^{-1}(x)) = (1/(1+x), x/(1+x))$.

This time, $(1/(1+x), x/(1+x))_{n,k} = [x^{n-k}](1+x)^{-(k+1)} = (-)^{n-k}\binom{n}{k}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & 0 & \cdots \\ -1 & 3 & -3 & 1 & 0 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

Partial sum theorem

If $g(x)$ is the generating function of some sequence $\{g_k\}_{k\geqslant 0}$ then

$$\boxed{\sum_{k=0}^{n}(p,q)_{n,k}g_k = [x^n]p(x)g(q(x))}$$

Proof: First we need $g(q(x)) = \sum_{k=0}^{\infty} g_k q(x)^k$ (where we must have $q_0 = 0$ to avoid an infinite constant term). Then

$$[x^n]p(x)g(q(x)) = [x^n]p(x)\sum_{k=0}^{\infty} g_k q(x)^k = \sum_{k=0}^{n}[x^n]p(x)q(x)^k g_k$$

Example: applying this to the Pascal's triangle Riordan array $(1/(1-x), x/(1-x))$ gives Euler's transformation:

$$\sum_{k=0}^{n}\binom{n}{k}g_k = [x^n]\frac{1}{1-x}\,g\left(\frac{x}{1-x}\right)$$

Example of using Euler's transformation to generate an identity

Put $g(x) = (1+x)^n$ so that $g_k = \binom{n}{k}$. Then

$$\sum_{k=0}^{n} \binom{n}{k} g_k = \sum_{k=0}^{n} \binom{n}{k}^2 = [x^n]\frac{1}{1-x}\left(1+\frac{x}{1-x}\right)^n = [x^n](1-x)^{-(n+1)}$$

Hence

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Properties of Riordan arrays (via the partial sum theorem)

Row sums (using $g = 1/(1-x)$ so $g_k = 1$):

$$\sum_{k=0}^{n} (p,q)_{n,k} = [x^n] \frac{p(x)}{1-q(x)}$$

Alternating row sums (using $g = 1/(1+x)$ so $g_k = (-1)^k$):

$$\sum_{k=0}^{n} (-)^k (p,q)_{n,k} = [x^n] \frac{p(x)}{1+q(x)}$$

Weighted row sums (using $g = x/(1-x)^2$ so $g_k = k$):

$$\sum_{k=0}^{n} k \ (p,q)_{n,k} = [x^n] \frac{p(x)q(x)}{(1-q(x))^2}$$

Diagonal sums (modified partial sum theorem)

$$[x^n]p(x)g(xq(x)) = [x^n]p(x)\sum_{k=0}^{\infty} g_k x^k q(x)^k = \sum_{k=0}^{n}[x^{n-k}]p(x)q(x)^k g_k$$
$$= \sum_{k=0}^{n}(p,q)_{n-k,k}g_k$$

So if $g_k = 1$, i.e., $g(x) = 1/(1-x)$,

$$\sum_{k=0}^{n}(p,q)_{n-k,k} = [x^n]\frac{p(x)}{1-xq(x)}$$

Example: Pascal's triangle (the $(1/(1-x), x/(1-x))$ Riordan array) gives

$$\sum_{k=0}^{n}\binom{n-k}{k} = [x^n]\frac{\frac{1}{1-x}}{1-\frac{x^2}{1-x}} = [x^n]\frac{1}{1-x-x^2} = F_{n+1}$$

where F_n is the nth Fibonacci number ($F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$).

The A-sequence

A (p, q) Riordan array is said to be *proper* iff $q_1 \neq 0$.

Proper Riordan arrays can also be obtained via the rule for generating the next row of the triangle from the row above. If

$$(p,q)_{n+1,k+1} = A_0(p,q)_{n,k} + A_1(p,q)_{n,k+1} + \cdots + A_j(p,q)_{n,k+j} + \cdots$$

(where p, q have yet to be determined) then $\{A_j\}_{j \geqslant 0}$ is known as the A-sequence of the Riordan array and the corresponding generating function $A(x)$ satisfies

$$q(x) = xA(q(x))$$

where q is the unique solution.

Example (Pascal's triangle): $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ and so $A(x) = 1 + x$. This gives $q = x/(1 - x)$.

The Z-sequence

We also need to know the values in the first column. These are again obtained from the row above, but this time via the Z-sequence $\{Z_j\}_{j\geqslant 0}$:

$$(p,q)_{n+1,0} = Z_0(p,q)_{n,0} + Z_1(p,q)_{n,1} + \cdots + Z_j(p,q)_{n,j} + \cdots$$

If $Z(x)$ is the generating function for the Z-sequence then

$$p(x) = \frac{p_0}{1 - xZ(q(x))}.$$

Example (Pascal's triangle): $\binom{n+1}{0} = \binom{n}{0}$ and so $Z(x) = 1$. Since $\binom{0}{0} = 1$ this gives $p(x) = 1/(1-x)$.

Coloured walks

Consider a ('positive') walk from 0 along the non-negative integers which in general is composed of right unit steps,'sur-place' steps (no change of position), and unit steps to the left. The respective types of steps can be of $r > 0$, $s \geqslant 0$, and $l \geqslant 0$ different varieties (or 'colours').

If $W_{n,k}$ is the number of possible walks containing n steps that end up a distance k from the origin then

$$W_{n+1,k+1} = rW_{n,k} + sW_{n,k+1} + lW_{n,k+2}$$

As $W_{n,n} = r^n \neq 0$ and so $q_1 \neq 0$, and we have an A-sequence $\{r, s, l\}$ (with generating function $A(x) = r + sx + lx^2$), $W_{n,k}$ are the entries in a Riordan array (p, q). Using $q(x) = xA(q(x))$ gives, if $l > 0$,

$$q(x) = \frac{1 - sx - \sqrt{(1 - sx)^2 - 4lrx^2}}{2lx}.$$

(The other root doesn't give a valid generating function.)

Coloured walks (Z-sequence and $p(x)$)

We must also have

$$W_{n+1,0} = sW_{n,0} + lW_{n,1}$$

from which $Z(x) = s + lx$. Using $p(x) = p_0/(1 - xZ(q(x)))$ gives

$$p(x) = \frac{1 - sx - \sqrt{(1 - sx)^2 - 4lrx^2}}{2lrx^2}$$

What do we want to know about these walks?

- the number of walks with n steps that return to the origin

  Answer: $W_{n,0} = (p, q)_{n,0} = p_n$

- the total number of walks with n steps

  Answer: the row sum, $\rho_n = \sum_{k=0}^{n} (p, q)_{n,k}$

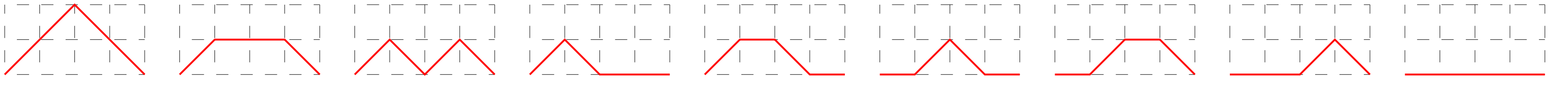
- the mean distance from the origin of all walks with n steps

  Answer: ω_n / ρ_n where ω_n is the weighted row sum, $\sum_{k=0}^{n} k \, (p, q)_{n,k}$

We already know how to calculate these quantities for a Riordan array (see results from the partial sum theorem).

Motzkin paths

A Motzkin path starts at the origin on the 2-d lattice $\mathbb{Z}^2$ and must finish on the x-axis. It must also never go below the x-axis. The allowed steps are $(1,1)$, $(1,0)$, and $(1,-1)$. There is a bijection between a Motzkin path and a $r = s = l = 1$ coloured positive walk that ends at the origin: $(1,1) \leftrightarrow$ right step; $(1,0) \leftrightarrow$ sur-place step; $(1,-1) \leftrightarrow$ left step.



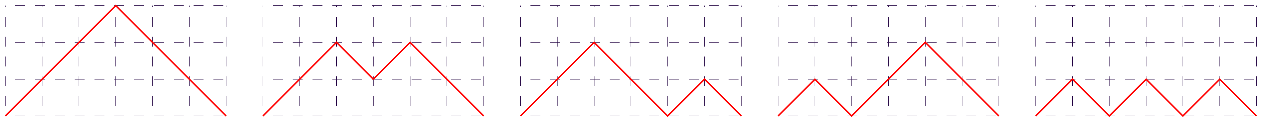
Putting $r = s = l = 1$ in the general expression for $p(x)$ for coloured positive walks gives the generating function

$$p(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} = 1+x+2x^2+4x^3+9x^4+21x^5+51x^6+\ldots$$

The number of Motzkin paths with n steps is $[x^n]p(x)$.

Dyck paths

A Dyck path is like a Motzkin path but the only allowed steps are $(1, 1)$ and $(1, -1)$. Hence the bijection is with coloured positive walks with $r = l = 1$, $s = 0$.



Hence the generating function is

$$p(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x^2} = 1 + x^2 + 2x^4 + 5x^6 + 14x^8 + 42x^{10} + \dots$$

The number of Dyck paths with n steps is $[x^n]p(x)$.

Note that all Dyck paths have an even number of steps.

(The sequence $1, 1, 2, 5, 14, 42, \dots$ is the Catalan numbers.)

Fundamental Theorem of Riordan Arrays (FTRA)

If

$$(p, q)\mathbf{g} = \mathbf{h},$$

where $\mathbf{g} = (g_0, g_1, g_2, \dots)^{\mathrm{T}}$, $\mathbf{h} = (h_0, h_1, h_2, \dots)^{\mathrm{T}}$, then

$$h(x) = p(x)g(q(x))$$

where $g(x) = g_0 + g_1 x + g_2 x^2 + \dots$ and $h(x) = h_0 + h_1 x + h_2 x^2 + \dots$.

Application of the FTRA to finding the average number of hills in a Dyck path with $2n$ steps

A hill in a Dyck path is an up step immediately followed by a down step where the up step must start from the x-axis. E.g., among the 5 Dyck paths with 6 steps, there are a total of 5 hills.

Let $(p, q)_{n,k}$ be the number of Dyck paths with $2n$ steps that have k hills. Then $p(x)$ must be the generating function for the number of Dyck paths with no hills (i.e., p_n is the number of Dyck paths with $2n$ steps that have 0 hills).

The generating function for the number of Dyck paths with 1 hill is $p(x)xp(x)$ (1 hill with a 0-hill Dyck path (possibly of 0 length) on either side). In general, the generating function for the number of Dyck paths with k hills is

$$p(x)\overbrace{xp(x)\cdots xp(x)}^{k \text{ times}} = p(x)(xp(x))^k$$

Thus $q(x) = xp(x)$.

Relation to the Catalan numbers

Let $C(x)$ be the generating function such that C_n is the total number of Dyck paths with $2n$ steps. (C_n are the Catalan numbers and $C(x) = (1 - \sqrt{1-4x})/2x$.)

Decomposing $C(x)$ into the cases with 0 hills, 1 hill, 2, hills, etc gives $C(x) = p(x) + xp(x)^2 + x^2p(x)^3 + \ldots$. Hence

$$C(x) = \frac{p(x)}{1 - xp(x)}$$

Counting the hills

Let $h(x)$ be the generating function such that h_n is the total number of hills present in all Dyck paths of length $2n$. Let $\mathbf{g} = (0, 1, 2, \dots)^{\mathrm{T}}$ and so $g(x) = x/(1-x)^2$. Then ordinary matrix multiplication tells us that

$$\mathbf{h} = (p, q)\mathbf{g}$$

and from the FTRA

$$\begin{aligned} h(x) &= p(x)g(q(x)) = p(x)q(x)/(1-q(x))^2 \\ &= xp(x)^2/(1-xp(x))^2 = xC^2(x) = C(x) - 1, \end{aligned}$$

where the final result is from $C(x)$ satisfying $xC^2(x) - C(x) + 1 = 0$. Hence, except when $n = 0$, $h_n = C_n$, and so for a given non-zero number of steps, the number of hills equals the number of Dyck paths, i.e., the average number of hills is 1.

References

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THANK YOU FOR YOUR ATTENTION