Hilbert C^*-modules

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The idea behind the introducing Hilbert C^*-modules is to generalize the concept of Hilbert spaces, replacing the field of scalars by a C^*-algebra.

• Let X be a (complex) vector space and A a C^*-algebra. If there is a bilinear map (module action)

$$\mathsf{X} \times A \to \mathsf{X}$$

$$(x, a) \mapsto x \cdot a$$

such that

$$\lambda(x \cdot a) = (\lambda x) \cdot a = x \cdot (\lambda a) \quad (x \in \mathsf{X}, a \in A, \lambda \in \mathbb{C}),$$

then X is called a (right) A-*module* and denoted by X_A.

Example 1. Let $M_n(A)$ be the vector space of all $n \times n$ matrices with entries in a C^*-algebra A. Then it is a (right) A-module with the action

$$(a_{i,j}) \cdot a := (a_{i,j}a).$$

Definition

An (right) inner product A-module is an (right) A-module together with a map $\langle\cdot,\cdot\rangle_A : \mathsf{X} \times \mathsf{X} \to A$ (A-valued inner product) such that

1. $\langle x, \lambda y + \mu z\rangle_A = \lambda\langle x, y\rangle_A + \mu\langle x, z\rangle_A$;
2. $\langle x, y \cdot a\rangle_A = \langle x, y\rangle_A a$;
3. $\langle x, y\rangle_A^* = \langle y, x\rangle_A$;
4. $\langle x, x\rangle_A \geq 0$ (a positive element of A);
5. $\langle x, x\rangle_A = 0 \iff x = 0$,

for all $x, y, z \in \mathsf{X}$ and $\lambda, \mu \in \mathbb{C}$.

- It follows from (1), (2), and (3) in above that

$$I := \operatorname{span}\{\langle x, y\rangle_A : x, y \in \mathsf{X}\}$$

is and ideal of A.

There is a norm on an inner product A-module X induced by its A-valued inner product $\langle \cdot, \cdot \rangle_A$:

$$\|x\|_A := \|\langle x, x \rangle_A\|^{1/2},$$

such that $\|x \cdot a\|_A \leq \|x\|_A \|a\|$ for all $x \in \mathsf{X}$ and $a \in A$. Now:

Definition

A Hilbert A-module is an inner product A-module X which is complete in the norm $\| \cdot \|_A$. It is called a full Hilbert A-module if the ideal

$$I := \mathrm{span}\{\langle x, y \rangle_A : x, y \in \mathsf{X}\}$$

is dense in A.

- Note that a left A-module, a left A-valued inner product ${}_A\langle \cdot, \cdot \rangle$, and a left Hilbert A-module can be defined in a similar way.

Example 2. Any Hilbert space H is actually a Hilbert module over $\mathbb{C}$ (Hilbert $\mathbb{C}$-module), with:

$$h \cdot \lambda := \lambda h \text{ (the usual sacalar multiplication); and}$$

$$\langle h, k\rangle_{\mathbb{C}} := \langle k|h\rangle \text{ (the usual Hilbert space inner product),}$$

for $h, k \in H$ and $\lambda \in \mathbb{C}$.

Example 3. Every C^*-algebra A can be viewed as a (full) Hilbert module over itself (the Hilbert A-module A_A), with

$$a \cdot b := ab \text{ (the usual multiplication in } A\text{); and}$$

$$\langle a, b\rangle_A := a^*b,$$

for $a, b \in A$. In this case, the norm $\|\cdot\|_A$ on A_A coincides with the C^*-norm of A as we have

$$\|a\|_A = \|\langle a, a\rangle_A\|^{1/2} = \|a^*a\|^{1/2} = \left(\|a\|^2\right)^{1/2} = \|a\|.$$

Example 4. Let H be a Hilbert space and $\mathcal{K}(H)$ the C^*-algebra of compact operators on H (as a C^*-subalgebra of bounded operators $B(H)$). For $x, y \in H$, let $x \otimes \overline{y}$ denote the rank-one operator

$$h \mapsto \langle h|y\rangle x = x \cdot \langle y, h\rangle_{\mathbb{C}}$$

on H. Now H is a full left Hilbert $\mathcal{K}(H)$-module with

$$T \cdot x := T(x) \text{ and } {}_{\mathcal{K}(H)}\langle x, y\rangle := x \otimes \overline{y}.$$

Note that the norm ${}_{\mathcal{K}(H)}\|\cdot\|$ of H coincides with the usual norm on H, because

$${}_{\mathcal{K}(H)}\|x\| = \|{}_{\mathcal{K}(H)}\langle x, x\rangle\|^{1/2} = \|x \otimes \overline{x}\|^{1/2} = \left(\|x\|^2\right)^{1/2} = \|x\|.$$

To see that ${}_{\mathcal{K}(H)}H$ is full, recall that the compact operators $\mathcal{K}(H)$ are indeed spanned by finite-rank operators, more precisely

$$\mathcal{K}(H) = \overline{\text{span}}\{x \otimes \overline{y} : x, y \in H\} = \overline{\text{span}}\{{}_{\mathcal{K}(H)}\langle x, y\rangle : x, y \in H\}.$$

Example 5. Let T be a locally compact Hausdorff space, and H a Hilbert space. Let

$$\mathsf{X} = C_0(T,H) := \{x : T \to H : x \text{ is continuous and } (t \mapsto \|x(t)\|) \in C_0(T)\}.$$

Then X is a Hilbert $C_0(T)$-module with

$$(x \cdot f)(t) := f(t)x(t) \quad \text{and} \quad \langle x,y\rangle_{C_0(T)}(t) := \langle y(t)|x(t)\rangle,$$

where $x, y \in \mathsf{X}$ and $f \in C_0(T)$. We have

$$\begin{aligned} \|x\|^2_{C_0(T)} = \|\langle x,x\rangle_{C_0(T)}\| &= \sup_{t\in T}|\langle x,x\rangle_{C_0(T)}(t)| \\ &= \sup_{t\in T}|\langle x(t)|x(t)\rangle| \\ &= \sup_{t\in T}\|x(t)\|^2 \\ &= \left(\sup_{t\in T}\|x(t)\|\right)^2 = \|x\|^2_\infty. \end{aligned}$$

Therefore the norm $\|\cdot\|_{C_0(T)}$ agrees with the usual sup norm on $C_0(T,H)$. Moreover X is full, because...

the C^*-algebra $C_0(T)$, which can be viewed as a Hilbert module over itself, admits an approximate unit, and therefore

$$C_0(T) = \overline{\text{span}}\{\langle f, g\rangle_{C_0(T)} : f, g \in C_0(T)\} = \overline{\text{span}}\{\overline{f}g : f, g \in C_0(T)\}.$$

But if we take $x, y \in \mathsf{X} = C_0(T, H)$ such that $x(t) := f(t)h$ and $y(t) := g(t)h$, where $h \in H$ with $\|h\| = 1$, then $\langle x, y\rangle_{C_0(T)} = \overline{f}g$. Because

$$\langle x, y\rangle_{C_0(T)}(t) = \langle g(t)h | f(t)h\rangle = g(t)\overline{f(t)}\langle h|h\rangle = \overline{f(t)}g(t) = (\overline{f}g)(t).$$

Therefore $C_0(T) = \overline{\text{span}}\{\langle x, y\rangle_{C_0(T)} : x, y \in \mathsf{X}\}$.

Example 6. (Direct Sums of Hilbert modules) Let X and Y be Hilbert A-modules. Then $Z = X \oplus Y := \{(x, y) : x \in X, y \in Y\}$ is a Hilbert A-module with

$$(x, y) \cdot a := (x \cdot a, y \cdot a) \quad \text{and;}$$

$$\langle (x_1, y_1), (x_2, y_2)\rangle_A := \langle x_1, x_2\rangle_A + \langle y_1, y_2\rangle_A.$$

Example 7. Let A be a C^*-algebra, and $\mathbb{N} := \{0, 1, 2, 3, ...\}$. Then

$$\ell^2(\mathbb{N}, A) = \{f : \mathbb{N} \to A : \sum_{n=0}^{\infty} f(n)^* f(n) \ \text{ converges in } A\}$$

is a (full) Hilbert A-module with

$$(f \cdot a)(n) := f(n)a \ \text{ and } \ \langle f, g\rangle_A := \sum_{n=0}^{\infty} f(n)^* g(n).$$

Note that $\ell^2(\mathbb{N}, A)$ can be identified with the *tensor product* $\ell^2(\mathbb{N}) \otimes A$.

Proposition (The Cauchy-Schwarz inequality)

If X is an inner product A-module, then

$$\langle x, y\rangle_A^* \langle x, y\rangle_A \le \|\langle x, x\rangle_A\| \langle y, y\rangle_A \ \text{ for all } x, y \in X,$$

Adjointable operators on Hilbert C^*-modules

Recall that the bounded operators on Hilbert spaces are automatically *adjointable*. But we are going to see that this is not true for operators on Hilbert C^*-modules. More precisely, Adjointable operators on Hilbert C^*-modules are bounded but a bounded operator on a Hilbert C^*-module may not be adjointable.

Definition

Let X and Y be Hilbert A-modules, where A is a C^*-algebra. A map $T : X \to Y$ is *adjointable* if there is a map $T^* : Y \to X$ such that

$$\langle T(x), y \rangle_A = \langle x, T^*(y) \rangle_A \text{ for all } x \in X, y \in Y.$$

Theorem

If a map $T : X \to Y$ is adjointable, then it is a bounded linear A-module map from X to Y.

Note that A-*module map* means that T is A-linear, which preserves the module action:

$$T(x \cdot a) = T(x) \cdot a \text{ for all } x \in X, a \in A.$$

Example 8. (A bounded A-linear operator on a Hilbert A-module which is **NOT** adjointable.)

Suppose $A = C([0,1])$, and let $J = \{f \in A : f(0) = 0\}$. Then A and J are Hilbert A-modules. Now take $X := A \oplus J$, and define $T : X \to X$ by $T(f,g) = (g,0)$. Then one can see that T is bounded such that $\|T\| = 1$ and A-linear. Assume that T has an adjoint T^* satisfying $\langle T(x), y\rangle_A = \langle x, T^*(y)\rangle_A$. If $(f,g) := T^*(1,0)$, then for all $(h,k) \in X$

$$\begin{aligned} \bar{k} = \langle (k,0),(1,0)\rangle_A &= \langle T(h,k),(1,0)\rangle_A \\ &= \langle (h,k), T^*(1,0)\rangle_A \\ &= \langle (h,k),(f,g)\rangle_A \\ &= \bar{h}f + \bar{k}g. \end{aligned}$$

So we must have $f \equiv 0$ and $g \equiv 1$, which contradicts $g(0) = 0$. Thus T cannot be adjointable.

Definition

If X and Y are Hilbert A-modules, then the set of all adjointable operators from X to Y is denoted by $\mathcal{L}(X,Y)$. For $\mathcal{L}(X,X)$, we simply write $\mathcal{L}(X)$.

One can see that if $T \in \mathcal{L}(X)$, then T^* is unique, $T^* \in \mathcal{L}(X)$, and $(T^*)^* = T$.

Also $\mathcal{L}(X)$ is in fact a subalgebra of the Banach algebra $B(X)$ of bounded operators on X, and $T \mapsto T^*$ is an involution on $\mathcal{L}(X)$. We indeed have:

Theorem

If X is a Hilbert A-module, then $\mathcal{L}(X)$ is a C^*-algebra with respect to the operator norm.

Example 9. Consider $\mathcal{L}(\ell^2(\mathbb{N}, A))$, the C^*-algebra of adjointable operators on the Hilbert A-modules $\ell^2(\mathbb{N}, A)$. Then define the map $S : \ell^2(\mathbb{N}, A) \to \ell^2(\mathbb{N}, A)$ by

$$S(f(0), f(1), f(2), ...) := (0, f(0), f(1), f(2), ...) \quad f \in \ell^2(\mathbb{N}, A)$$

Then $S \in \mathcal{L}(\ell^2(\mathbb{N}, A))$ such that

$$S^*(f(0), f(1), f(2), ...) = (f(1), f(2), ...).$$

Also it is easy to that $S^*S = I$ but $SS^* \neq I$, which means S is a nonunitary isometry.

Adjointable operators on Hilbert C^*-modules

Moreover if α is an endomorphism of A, then the map $\pi_\alpha : A \to \mathcal{L}(\ell^2(\mathbb{N}, A))$ defined by

$$\begin{aligned}\pi_\alpha(a)(f(0), f(1), f(2), ...) &= (\alpha^0(a)f(0), \alpha^1(a)f(1), \alpha^2(a)f(2), ...) \\ &= (af(0), \alpha(a)f(1), \alpha^2(a)f(2), ...)\end{aligned}$$

is an injective $*$-homomorphism of C^*-algebra. So in fact, π_α is a faithful representation of A on the Hilbert A-modules $\ell^2(\mathbb{N}, A)$.

Thank you