The Nica-Toeplitz algebras of abelian lattice ordered groups as full corners in group crossed products

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The research area of interest

My research interest lies in the area of operator algebras (and functional analysis), where in particular, I am working on the representation theory of C^*-dynamical systems and the (ideal) structure of C^*-algebras arising from C^*-dynamical systems.

Currently, I am working on the semigroup C^*-dynamical systems of quasi-lattice ordered groups and their associated Nica-Toeplitz algebra.

Quasi-lattice ordered groups

Let P be a generating subsemigroup of a (nonabelian) discrete group G such that $P \cap P^{-1} = \{e\}$. For every $x, y \in G$, define

$$x \leq_{\mathrm{rt}} y \iff yx^{-1} \in P \iff y \in Px.$$

Then, $\leq_{\mathrm{rt}}$ is a partial order on G.

Definition (Nica, 1992)

(G, P) is *(right) quasi-lattice ordered* if each pair $x, y \in G$ with a common upper bound in P has a least upper bound $x \vee_{\mathrm{rt}} y$ in P.

Note that $\leq_{\mathrm{rt}}$ is therefore right invariant, meaning that, if $x \leq_{\mathrm{rt}} y$, then $xz \leq_{\mathrm{rt}} yz$ for every $z \in G$. Also, one may define

$$x \leq_{\mathrm{lt}} y \iff x^{-1}y \in P \iff y \in xP,$$

and hence, the *left quasi-lattice order* in a similar way.

Some examples of Quasi-lattice ordered groups

Example 1. $(\mathbb{Z}, \mathbb{N})$ with $\leq$ is the usual total order, where $\mathbb{N} = \{0, 1, 2, 3, ...\}$ and $m \vee n = \max\{m, n, 0\}$ for every $m, n \in \mathbb{Z}$.

Example 2. (Direct sum) Assume that $k > 1$ is a natural number. Then, in $(\mathbb{Z}^k, \mathbb{N}^k)$, we have

$$(m_1, m_2, ..., m_k) \leq (n_1, n_2, ..., n_k) \Longleftrightarrow m_i \leq n_i.$$

Therefore,

$$(m_1, m_2, ..., m_k) \vee (n_1, n_2, ..., n_k) = (s_1, s_2, ..., s_k),$$

where $s_i = \max\{m_i, n_i, 0\}$.

Example 3. Consider $(\mathbb{Q}_+^*, \mathbb{N}^*)$ with multiplication, where $\mathbb{N}^* = \{1, 2, 3, ...\}$. Then, for every $r, s \in \mathbb{Q}_+^*$,

$$r \leq s \Longleftrightarrow sr^{-1} = (\frac{s}{r}) \in \mathbb{N} \Longleftrightarrow r|s.$$

Like: $\frac{3}{5} \leq \frac{6}{5}$, as $(\frac{6}{5})(\frac{5}{3}) = 2 \in \mathbb{N}$.

So, note that, here

$$r \vee s = \text{the lowest common multiple in } \mathbb{N}^*.$$

Like:

$$(\frac{7}{3}) \vee (\frac{5}{11}) = 35;$$

$$(\frac{2}{5}) \vee (\frac{3}{5}) = 6.$$

Some examples of Quasi-lattice ordered groups

Note that, in fact, **Example 2** and **Example 3** are (abelian) lattice ordered group. Recall that a lattice ordered group G is a partially ordered group such that each pair $x, y \in G$ has a unique $\sup$ and a unique $\inf$ in G.

Like, in $(\mathbb{Z}^k, \mathbb{N}^k)$:

$$\sup\{(m_1, m_2, ..., m_k), (n_1, n_2, ..., n_k)\} = (s_1, s_2, ..., s_k),$$

where $s_i = \max\{m_i, n_i\}$, and

$$\inf\{(m_1, m_2, ..., m_k), (n_1, n_2, ..., n_k)\} = (r_1, r_2, ..., r_k),$$

where $r_i = \min\{m_i, n_i\}$.

The semigroup C^*-dynamical system and its Nica-Toeplitz algebra

Now, we consider the C^*-dynamical system (A, P, α) consisting of

• a C^*-algebra A,

• a generating subsemigroup P of a (nonabelian) discrete group G such that (G, P) is quasi-lattice ordered, and

• an action $\alpha : P \to \mathrm{End}(A)$ of P by (extendible) endomorphisms of A.

Theorem (Fowler, 2002)

He showed that there is a universal C^*-algebra $\mathcal{T}_{cov}(A \times_\alpha P)$ associated to the system (A, P, α) generated by a universal **Nica-Toeplitz covariant representation** *of the system, such that its nondegenerate representations are in a bijective correspondence with the Nica-Toeplitz covariant representation of (A, P, α).*

$\mathcal{T}_{\mathrm{cov}}(A \times_\alpha P)$ is called the *Nica-Toeplitz algebra*.

Undergoing project...

So, in my current research work, I am considering the C^*-dynamical system (A, P, α), where (G, P) is any abelian lattice ordered group.

I have shown that:

Theorem

$\mathcal{T}_{cov}(A \times_\alpha P)$ is a **full corner** *in a classical crossed product C^*-algebra $\mathcal{B} \times_\beta G$ by the group G associated with a classical system $(\mathcal{B}, G, \beta)$.*

Thus, $\mathcal{T}_{\mathrm{cov}}(A \times_\alpha P)$ and $\mathcal{B} \times_\beta G$ are **Morita equivalent**, and hence, they have the same ideal structure and representation theory. Consequently, we can import the information from the well-established theory of crossed products by groups to understand the structure of $\mathcal{T}_{\mathrm{cov}}(A \times_\alpha P)$.

A quick recall

Definition

A *(right) Nica partial-isometric representation* of (G, P) on a Hilbert space H is a map $V : P \to B(H)$ such that each $V_x := V(x)$ is a partial isometry and $V_x V_y = V_{xy}$, and which satisfies

$$V_x^* V_x V_y^* V_y = \begin{cases} V_{x \vee_{\mathrm{rt}} y}^* V_{x \vee_{\mathrm{rt}} y} & \text{if } x \vee_{\mathrm{rt}} y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Example 4. For $(\mathbb{Q}_+^*, \mathbb{N}^*)$, take $V : \mathbb{N}^* \to B(\ell^2(\mathbb{N}^*) \otimes H)$ defined by

$$(V_s f)(t) = f(ts)$$

for every $f \in \ell^2(\mathbb{N}^*) \otimes H \simeq \ell^2(\mathbb{N}^*, H)$.

A quick recall

Definition

A *Nica-Toeplitz covariant representation* of (A, P, α) is a pair (π, V) consisting of a non-degenerate representation π of A and a (right) Nica partial-isometric representation V of P on a Hilbert space H, such that

1. $\pi(\alpha_s(a)) = V_s\pi(a)V_s^*$;
2. $V_s^*V_s\pi(a) = \pi(a)V_s^*V_s$.

for all $a \in A$ and $s \in P$.

A quick recall

Theorem (Fowler, 2002)

He showed that there is a universal C^*-algebra $\mathcal{T}_{cov}(A \times_\alpha P)$ associated to the system (A, P, α) generated by a universal **Nica-Toeplitz covariant representation** (i_A, i_P) *of the system (A, P, α) with i_A injective, such that there is a bijective correspondence $(\pi, V) \mapsto \pi \times V$ between the Nica-Toeplitz covariant representation of (A, P, α) and the nondegenerate representations of $\mathcal{T}_{cov}(A \times_\alpha P)$.*

$\mathcal{T}_{\mathrm{cov}}(A \times_\alpha P)$ is called the *Nica-Toeplitz algebra*.

Undergoing project...

Consider the C^*-dynamical system (A, P, α), where (G, P) is any abelian lattice ordered group. For every $s \in G$, define a map $\phi_s : A \to \ell^\infty(G, A)$ by

$$\phi_s(a)(x) = \begin{cases} \alpha_{xs^{-1}}(a) & \text{if } s \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

Then, let $\mathcal{B}$ be the C^*-subalgebra of $\ell^\infty(G, A)$ generated by $\{\phi_s(a) : s \in G, a \in A\}$. In fact, we have

$$\mathcal{B} = \overline{\operatorname{span}}\{\phi_s(a) : s \in G, a \in A\},$$

which contains

$$\mathcal{J} = \overline{\operatorname{span}}\{\phi_s(a) - \phi_t(\alpha_{ts^{-1}}(a)) : s < t \in G, a \in A\}$$

as an essential ideal.

There is an action β of G by automorphisms on $\mathcal{B}$ induced by the shift on $\ell^\infty(G, A)$, such that $\beta_t \circ \phi_s = \phi_{ts}$ for all $s, t \in G$. Thus, we obtain a classical dynamical system $(\mathcal{B}, G, \beta)$.

Let $(\mathcal{B} \times_\beta G, j_\mathcal{B}, j_G)$ be the group crossed product C^*-algebra associated to the system $(\mathcal{B}, G, \beta)$. Since $\mathcal{J}$ is a β-invariant essential ideal of $\mathcal{B}$, $\mathcal{J} \times_\beta G$ sits in $\mathcal{B} \times_\beta G$ as an essential ideal.

Theorem

Let (A, P, α) be a C^*-dynamical system, where (G, P) is any abelian lattice ordered group. Then, $\mathcal{T}_{cov}(A \times_\alpha P)$ is a **full corner** *in the classical crossed product C^*-algebra $\mathcal{B} \times_\beta G$.*

Undergoing project...

Furthermore, there is an (essential) ideal

$$\mathcal{I} := \overline{\text{span}}\{i_P(x)^* i_A(a)(1 - i_P(s)^* i_P(s)) i_P(y) : a \in A, x, y, s \in P\}$$

of $\mathcal{T}_{\text{cov}}(A \times_\alpha P)$, which (**conjecture=**) is the kernel of the natural surjective homomorphism $\varphi : \mathcal{T}_{\text{cov}}(A \times_\alpha P) \to A \times_\alpha^{\text{iso}} P$. More precisely, we have a short exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow \mathcal{T}_{\text{cov}}(A \times_\alpha P) \xrightarrow{\varphi} A \times_\alpha^{\text{iso}} P \longrightarrow 0,$$

where $\ker\varphi = \mathcal{I}$. Then:

Lemma

The ideal $\mathcal{I}$ of $\mathcal{T}_{cov}(A \times_\alpha P)$ is a **full corner** *in the (essential) ideal $\mathcal{J} \times_\beta G$ of $\mathcal{B} \times_\beta G$.*

- For any system $(A, \mathbb{N}^2, \alpha)$ associated with $(\mathbb{Z}^2, \mathbb{N}^2)$, we have:

Theorem

$$\mathcal{J} \simeq (C_0(\mathbb{Z}) \otimes D_\delta) \oplus (C_0(\mathbb{Z}) \otimes D_\gamma),$$

which contains $C_0(\mathbb{Z}^2) \otimes A \simeq C_0(\mathbb{Z}) \otimes C_0(\mathbb{Z}) \otimes A$ *as an essential ideal.*

And, hence

Theorem

The ideal $\mathcal{I}$ of $\mathcal{T}_{cov}(A \times_\alpha \mathbb{N}^2)$ is a full corner in the algebra

$$[\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\delta \times_\tau \mathbb{Z})] \oplus [\mathcal{K}(\ell^2(\mathbb{Z})) \otimes (D_\gamma \times_\tau \mathbb{Z})],$$

which contains the algebra $\mathcal{K}(\ell^2(\mathbb{Z}^2)) \otimes A \simeq \mathcal{K}(\ell^2(\mathbb{Z})) \otimes \mathcal{K}(\ell^2(\mathbb{Z})) \otimes A$ as an essential ideal.

Note that, D_δ and D_γ are two C^*-subalgebra of $\ell^\infty(\mathbb{Z}, A)$, which are actually the algebra $\mathcal{B}$ associated with the systems $(A, \mathbb{N}, \delta)$ and $(A, \mathbb{N}, \gamma)$, where

$$\delta_n := \alpha_{(0,n)} \text{ and } \gamma_n := \alpha_{(n,0)}$$

for every $n \in \mathbb{N}$.

• As an another example, it is natural to consider the abelian lattice-ordered group $(\mathbb{Q}_+^*, \mathbb{N}^*)$. At the time being, I do not know much about this case. Some results regarding this case are left to work on, which may come out in my subsequent works in future.

• Note that, for the totally ordered abelian group $(\mathbb{Z}, \mathbb{N})$, we completely fall into the context of

"**S. Zahmatkesh**, *The Partial-isometric crossed products by semigroups of endomorphisms are Morita-Equivalent to crossed products by groups*, New Zealand J. Math. 47 (2017), 121-139."

This is due to the fact that the Nica-Toeplitz algebra $\mathcal{T}_{\rm cov}(A \times_\alpha \mathbb{N})$ is the partial-isometric crossed product $A \times_\alpha^{\rm piso} \mathbb{N}$ introduced by Lindiarni and Raeburn.

Undergoing project...

Next, we consider any system (A, P, α), where (G, P) is any abelian lattice ordered group, the action α of P on A is given by automorphism. Then, we see that, we can have better descriptions/identifications for the C^*-algebras involved in this research work in terms of tensor product. In particular, for any automorphic system $(A, \mathbb{N}^2, \alpha)$, we expect some identifications with familiar terms.

Thank You!