# Symmetric Functions and Their Identities from Character Tables

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# Robinson-Schensted Algorithm

**Definition**

A **partition** of $n$ is a sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$$

of positive integers where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

We write $\lambda \vdash n$ if $\lambda$ is a partition of $n$.

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**Definition**

A **standard tableau** T is a partition $\lambda$ of $n$ which you insert the numbers $1, 2, \dots, n$ into each square exactly once and the numbers increase along each row and column.

# Robinson-Schensted Algorithm (Cont.)

Let $f^\lambda$ be the number of standard tableau of shape $\lambda$.
Robinson-Schensted Algorithm [Rob 38, Sch 61] provides a bijective proof of the identity

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!.$$

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**Theorem (Robinson-Schensted)**

*The following map is a bijection between permutation $\pi \in S_n$ and the pair of standard tableau of the same shape $(P, Q)$.*

This algorithm is generally known as the **bumping algorithm**.
Here I will give an example for which $\pi = (4, 2, 3, 6, 5, 1, 7)$ maps to.

# The Ring of Symmetric Functions

In fact the previous identity is only a small part of the much bigger structure from the theory of symmetric functions (invariant theory).

**Definition**

A polynomial is **symmetric** if it is invariant under permuting the variables.

The symmetric polynomials form a subring of $\mathbb{Z}[x_1, x_2, .., x_m]$.

# The Ring of Symmetric Functions (Cont.)

There are different types of symmetric functions:

**1. Monomial Symmetric Functions,** $m_\lambda$

For each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$

$$x^\alpha = x_1^{\alpha_1} \ldots x_m^{\alpha_m}.$$

The polynomial

$$m_\lambda(x_1, x_2, \ldots, x_n) = \sum x^\alpha$$

summed over all distinct permutations $\alpha$ of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$.

# The Ring of Symmetric Functions (Cont.)

**2. Elementary Symmetric Functions,** $e_r$

$$e_r = \sum x_{i_1} x_{i_2} \dots x_{i_r} = m_{(1^r)}.$$

where $i_1 < i_2 < \cdots < i_r$.
The generating function for the $e_r$ is

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t).$$

# The Ring of Symmetric Functions (Cont.)

**3. Complete Symmetric Functions,** $h_r$

$$h_r = \sum_{|\lambda|=r} m_\lambda.$$

The generating function for the $h_r$ is

$$H(t) = \sum_{r\geq 0} h_r t^r = \prod_{i\geq 1} (1 - x_i t)^{-1}.$$

As a result, we have that

$$H(t)E(-t) = 1$$

or equivalently,

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0, \qquad \text{for } n \geq 1.$$

# The Ring of Symmetric Functions (Cont.)

**4. Power Sum Functions,** $p_r$

For $r \geq 1$,

$$p_r = \sum x_i^r = m_{(r)}.$$

The generating function for the $p_r$ is

$$P(t) = \sum_{r \geq 1} p_r t^{r-1} = \frac{H'(t)}{H(t)}.$$

Likewise,

$$P(-t) = \frac{E'(t)}{E(t)}.$$

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Therefore,

$$nh_n = \sum_{r=1}^{n} p_r h_{n-r}, \qquad ne_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r}.$$

# The Ring of Symmetric Functions (Cont.)

**5. Schur Functions,** $s_\lambda$

Let $a_\lambda = \det(x_i^{\lambda_j})_{1\le i,j\le n}$

Let $\delta = (n-1, n-2, \ldots, 1, 0)$.

The famous Vandermonde determinant:

$$a_\delta = \det(x_i^{n-j}) = \prod_{1\le i<j\le n} (x_i - x_j).$$

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**Definition**

The **Schur** function $s_\lambda$ is defined by

$$s_\lambda = s_\lambda(x_1, x_2, \ldots, x_n) = a_{\lambda+\delta}/a_\delta = \det(x_i^{\lambda_j+n-j})/\det(x_i^{n-j}).$$

(Why Schur function is a polynomial?)

# The Ring of Symmetric Functions (Cont.)

We can present Schur function as a determinant of other symmetric polynomials.

**Theorem (Jacobi-Trudi Identity)**

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \le i,j \le n} \qquad \textit{where } n \geq \textit{length}(\lambda).$$

Similarly,

$$s_\lambda = \det(e_{\lambda'_i - i + j}),$$

where $\lambda'$ is the transpose of $\lambda$.

# Schur Functions and The Counting Problem

In this first section, we mentioned about the standard tableau.
However the numbers to be put in the squares do not need to be distinct.

**Definition**

A generalized tableau is *semistandard* if its rows weakly increase and its columns strictly increase.

# Schur Functions and The Counting Problem

In this first section, we mentioned about the standard tableau. However the numbers to be put in the squares do not need to be distinct.

**Definition**

A generalized tableau is *semistandard* if its rows weakly increase and its columns strictly increase.

Important connection between Schur functions and counting tableau:

**Theorem**

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T,$$

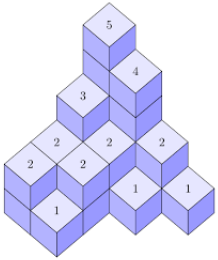
*where the sum is over all semistandard tableau $T$ of shape $\lambda$.*

# Connection to Plane Partition

**Definition**

The plane partition of shape $\lambda$ is a mapping $\phi$ from $\lambda$ to the positive integers such that $\phi(x_1) \geq \phi(x_2)$ whenever $x_2$ lies below or to the right of $x_1$.

Example:



# Connection to Plane Partition

**Definition**

Let $pp(n)$ denote the total number of plane partitions of $n$.

**Theorem (MacMahon's plane partition formula, 1916)**

*The generating function for the number of plane partitions is*

$$\sum_{n=0}^{\infty} pp(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}.$$

It is a well known story that it took him 20 years(!) to prove this theorem. However with the help of theorem from Schur function, this theorem is not too hard to be proved anymore.

# The proof of MacMahon's Plane Partition Theorem

The proof has been much better understood and can be fit into only 2 pages of the Macdonald's text book.

1. Robinson-Schensted Algorithm (Bumping Algorithm)
2. The Ring of Symmetric Functions
3. Schur Functions and The Counting Problem
4. Kostka Numbers, Characters and Experimental Mathematics

# Kostka Numbers

**Definition**

**Kostka numbers,** $K_{\lambda,\mu}$ is the number of semistandard tableaux of shape $\lambda$ and weight $\mu$. ($|\lambda| = |\mu|$).

Example: $K_{\lambda,(1,1,\ldots,1)} = f^{\lambda} =$ number of standard tableaux of shape $\lambda$.
$K_{(n),\mu} = 1$.
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$K_{(n),\mu} = 1$.
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We saw in the first section that

$$\sum_{\lambda \vdash n} (f^{\lambda})^2 = n!.$$

But this is only the tip of the iceberg!

# Kostka Numbers

If one restricts the latter sum to go over partition with most a fixed number of row, one gets many interesting sequences.

$$a_r(n) = \sum_{\substack{\lambda \vdash n \\ length(\lambda) \leq r}} (f^\lambda)^2.$$

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Analogous sequences, for the straight sums are also interesting.

$$b_r(n) = \sum_{\substack{\lambda \vdash n \\ length(\lambda) \le r}} f^{\lambda}.$$

Let's experiment this!!

# Characters

Kostka numbers can be viewed in the language of linear algebra as follows:

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A character $\chi^\lambda_\mu$ of $S_n$ can be defined similarly,

**Definition**

$$p_\mu = \sum_\lambda \chi^\lambda_\mu s_\lambda.$$

Hence, $\chi^\lambda_\mu$ equals to the coefficient of $x^{\lambda+\delta}$ in $a_\delta p_\mu$.
(There is no explicit way to describe $\chi^\lambda_\mu$).

# The Search for Identities of Characters

We can do the search for identities of these characters similar to what we did for Kostka numbers earlier.

- Sum overall all the partition with at most $r$ rows:

$$\psi_{r,\mu}^{(p)}(n) = \sum_{\substack{\lambda \vdash n \\ length(\lambda) \leq r}} (\chi_{\mu}^{\lambda})^{p}.$$

- Sum overall all the partition of the hook shape:

$$\phi_{\mu}^{(p)}(n) = \sum_{i=1}^{n} (\chi_{\mu}^{(i,1^{n-i})})^{p}.$$

# References


- I.G. Macdonald, *Symmectric Functions and Hall Polynomials, 2nd ed., Clarendon Press, Oxford, 1995.*
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- B. Sagan, *The Symmetric Group, 2nd ed., Springer, 2000.*