Lecture on Sylvester Matrices and Resultant of Polynomials

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Introduction to Sylvester's Matrix

- The Sylvester matrix is a matrix associated with two polynomials.
- Named after James Joseph Sylvester, an English mathematician.
- Given two polynomials $g(x)$ and $h(x)$ of degree m and n respectively, the Sylvester matrix $S(g, h)$ is an $(m+n) \times (m+n)$ matrix.

Introduction to Sylvester's Matrix (Part 2)

- The Sylvester matrix is constructed using the coefficients of the polynomials.
- It is a particular kind of resultant matrix and contains all the coefficients of the polynomials.
- It plays a significant role in solving polynomial equations and is a useful tool in algebraic geometry.

The Resultant of Two Polynomials (Part 1)

- The resultant of two polynomials is a scalar that gives information about the roots of the polynomials.
- It is denoted as $Res(g, h)$, where g and h are the two polynomials.
- If $Res(g, h) = 0$, this means that g and h have a common root. If $Res(g, h) \neq 0$, g and h have no common root.

The Resultant of Two Polynomials (Part 2)

- The resultant of two polynomials can also be computed from their Sylvester matrix.
- Specifically, $Res(g, h)$ is equal to the determinant of the Sylvester matrix of g and h, denoted $S(g, h)$.
- This connection between resultants and determinants of Sylvester matrices provides a powerful computational tool for working with polynomials.

Computation of Sylvester Matrix (Part 1)

- Given two polynomials $g(x)$ and $h(x)$, we can construct the Sylvester matrix $S(g, h)$.
- $g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$,
- $h(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$.
- The Sylvester matrix is an $(m+n) \times (m+n)$ matrix, constructed from the coefficients of the polynomials.

Computation of Sylvester Matrix (Part 2)

- To build $S(g, h)$, we create two types of rows: those based on $g(x)$ and those based on $h(x)$.
- Rows based on $g(x)$ are: $[a_m, a_{m-1}, \ldots, a_0, 0, \ldots, 0]$.
- Rows based on $h(x)$ are: $[b_n, b_{n-1}, \ldots, b_0, 0, \ldots, 0]$.
- Each subsequent row is a right shift of the previous row. If the shift moves a coefficient off the row, a zero is added at the other end.

Computation of Sylvester Matrix (Part 3)

- Let's consider an example.
- For $g(x) = x^3 + 2x^2 + 3$ and $h(x) = 5x^2 + x + 2$, the Sylvester matrix $S(g, h)$ is

$$\begin{bmatrix} 3 & 2 & 0 & 3 & 0 \\ 0 & 3 & 2 & 0 & 3 \\ 5 & 1 & 2 & 0 & 0 \\ 0 & 5 & 1 & 2 & 0 \\ 0 & 0 & 5 & 1 & 2 \end{bmatrix}$$

Applications of Sylvester's Matrix and Resultant

- Sylvester's Matrix and the Resultant have numerous applications in many areas of mathematics and computer science.
- They are used in algorithms for polynomial greatest common divisor (GCD) computation.
- They are also used for solving systems of polynomial equations, in algebraic geometry, and in the study of polynomial roots.
- In computer graphics and robotics, they are used for solving problems involving rotations and translations.

Computation of Resultant using Sylvester's Matrix (Part 1)

- As discussed earlier, the Resultant of two polynomials can be computed using their Sylvester matrix.
- Specifically, the Resultant $Res(g, h)$ is equal to the determinant of the Sylvester matrix of g and h, denoted $S(g, h)$.
- To calculate the Resultant using the Sylvester matrix, we simply need to compute the determinant of the matrix.

Computation of Resultant using Sylvester's Matrix (Part 2)

- Let's consider the example from the earlier slide.
- For $g(x) = x^3 + 2x^2 + 3$ and $h(x) = 5x^2 + x + 2$, the Sylvester matrix $S(g, h)$ is

$$\begin{bmatrix} 3 & 2 & 0 & 3 & 0 \\ 0 & 3 & 2 & 0 & 3 \\ 5 & 1 & 2 & 0 & 0 \\ 0 & 5 & 1 & 2 & 0 \\ 0 & 0 & 5 & 1 & 2 \end{bmatrix}$$

- By computing the determinant of this matrix, we obtain the Resultant $Res(g, h) = 944 = 2^4 \cdot 59$.

Case Studies (Part 3)

- The final case study will consider an application in algebraic geometry, specifically the problem of finding the intersection points of two curves.
- The intersection points correspond to the common roots of the polynomials that define the curves.
- Using Sylvester's Matrix and the Resultant, we can effectively solve for these intersection points, providing a powerful tool for studying geometric shapes.

Case Studies: Example in Algebraic Geometry

- Let's take the two polynomials $g(x) = x^2 + 2x + 1$ and $h(x) = x^2 - 1$ which represent two curves in the plane.
- The Sylvester matrix of these polynomials is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

- The determinant of this matrix is 0, which means the two polynomials have a common root. Indeed, they both intersect at $x = -1$.

Case Studies (2/2) - Mathematical proofs using Sylvester's matrix and resultant

- Beyond their practical applications, Sylvester's Matrix and the Resultant are also extensively used in mathematical proofs.
- For instance, they can be used to prove that two polynomials have a common root if and only if their resultant is zero.
- In the context of algebraic geometry, this is a key step in proving Bezout's Theorem, which states that two algebraic curves intersect at a number of points equal to the product of their degrees.

Software and Tools: SageMath (1/3) - Introduction to SageMath and its uses in polynomial algebra

- SageMath is a free, open-source mathematics software system that combines the power of many existing open-source packages into a common Python-based interface.
- SageMath is capable of performing a wide range of mathematical computations, including those related to polynomial algebra.
- Specifically, SageMath provides functionality to perform operations on polynomials, compute Sylvester matrices and resultants, and solve polynomial equations.
- In the next slides, we will walk through some examples of how to use SageMath for computations related to Sylvester matrices and resultants.

Software and Tools: SageMath (2/3) - Demo: Computing Sylvester matrices using SageMath

```
# Define a polynomial ring in x over the integers
R.<x> = ZZ[]

# Define two polynomials g and h
g = x^2 + 2*x + 1
h = x^2 - 1

# Compute the Sylvester matrix
S = g.sylvester_matrix(h)

# Print the Sylvester matrix
print(S)
```

Software and Tools: SageMath (3/3) - Demo: Computing the resultant of polynomials using SageMath

```
# Define a polynomial ring in x over the integers
R.<x> = ZZ[]

# Define two polynomials g and h
g = x^2 + 2*x + 1
h = x^2 - 1

# Compute the resultant of g and h
res = g.resultant(h)

# Print the resultant
print(res)
```

Conclusion (1/2) - Summary of main points

- Sylvester's Matrix and the Resultant are mathematical tools that provide an effective way to deal with polynomial equations, especially for finding common roots and computing GCDs.
- They have broad applications in various fields including computer science, robotics, and algebraic geometry.
- Despite certain limitations, these tools have proven to be valuable in many practical applications.
- The open-source software SageMath provides useful functionalities for performing calculations related to Sylvester matrices and resultants.

Good Reduction: Definition and Importance

- In arithmetic dynamics, a map has good reduction at a prime if it can be described by polynomials with coefficients in the ring of integers localized at that prime.
- The concept of good reduction is used to study the arithmetic properties of dynamical systems.
- Good reduction is a tool to investigate the behavior of dynamical systems under change of coordinates, scale changes, or more generally, change of model.
- The study of good reduction provides insights into the arithmetic and geometric structure of the dynamical system.

Defining Good Reduction in Rational Maps

Definition

A rational map $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with rational coefficients, has good reduction at a prime number p if we can multiply P and Q by an appropriate power of p to get polynomials $P'(x)$ and $Q'(x)$ with integer coefficients such that the resultant $\text{Res}(P', Q')$ of P' and Q' is not divisible by p.

Example of Good Reduction

Example

Consider the rational map $f(x) = \frac{x^2+\frac{1}{2}x}{x^2+2}$. We wish to check if it has good reduction at $p = 2$.
Multiply g and h by 2 to get $g'(x) = 2x^2 + x$ and $h'(x) = 2x^2 + 4$ with integer coefficients.
The resultant of $g'(x)$ and $h'(x)$ is $\text{Res}(g', h') = 72 = 2^3 \cdot 3^2$, which is divisible by 2 and 3.
Hence, the map $f(x)$ has a good reduction at prime $p \neq 2, 3$.

References

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